Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A short table of integrals is included on Page 2.

1. [**18 Points**] Solve the initial value problem: $y' = 4t^3y^2, \quad y(0) = 1/4$

   ▶ **Solution.** This equation is separable. After separating variables, it becomes $y^{-2}y' = 4t^3$, which in differential form is $y^{-2}\,dy = 4t^3\,dt$ and integration then gives $-y^{-1} = t^4 + C$, so that solving for y gives $y = -1/(t^4 + C)$. The initial condition $y(0) = 1/4$ means that $y = 1/4$ when $t = 0$. This gives $1/4 = -1/C$ so $C = -4$. Hence, the solution of the initial value problem is

   $$\boxed{y = \frac{-1}{t^4 - 4} = \frac{1}{4 - t^4}.}$$

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2. [**18 Points**] Find the general solution of: $y' - 3y = 5e^{3t} + e^t$

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = -3$ so that an integrating factor is given by $\mu(t) = e^{\int -3\,dt} = e^{-3t}$. Multiplication of the differential equation by the integrating factor gives

   $$e^{-3t}y' - 3e^{-3t}y = 5 + e^{-2t},$$

   and the left hand side is recognized (by choice of $\mu(t)$) as a perfect derivative:

   $$\frac{d}{dt}(e^{-3t}y) = 5 + e^{-2t}.$$

   Integration then gives

   $$e^{-3t}y = 5t - \frac{1}{2}e^{-2t} + C,$$

   where C is an integration constant. Multiplying through by e^{3t} gives

   $$\boxed{y = 5te^{3t} - \frac{1}{2}e^t + Ce^{-3t}.}$$

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3. [**18 Points**] Solve the initial value problem: $ty' + 4y = t, \quad y(1) = -1$

   ▶ **Solution.** This equation can be rewritten in standard form as

   $$y' + \frac{4}{t}y = 1.$$

This is linear with coefficient function $p(t) = 4/t$, so that an integrating factor is $\mu(t) = e^{\int (4/t)\,dt} = 4e^{\ln t} = t^4$. Multiplication of the differential equation by the integrating factor gives

$$t^4 y' + 4t^3 y = t^4,$$

and the left hand side is recognized (by choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(t^4 y) = t^4,$$

Integration then gives

$$t^4 y = \frac{1}{5}t^5 + C,$$

where C is an integration constant. Multiplying through by t^{-4} gives

$$y = \frac{1}{5}t + Ct^{-4}.$$

The initial condition $y(1) = -1$ implies $-1 = y(1) = 1/5 + C$ so that $C = -6/5$. Hence

$$\boxed{y = \frac{t}{5} - \frac{6}{5t^4}.}$$

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4. **[18 Points]** Solve the initial value problem: $(4t + 4y + 3) + (4t - 6y - 2)y' = 0$, $y(2) = 1$

▶ **Solution.** Letting $M(t,\, y) = 4t + 4y + 3$ and $N(t,\, y) = 4t - 6y - 2$, the equation can be written in the form $M + Ny' = 0$, and since

$$\frac{\partial M}{\partial y} = 4 = \frac{\partial N}{\partial t},$$

it follows that the equation is exact. Hence, we solve it by looking for a potential function $V(t,\, y)$ such that $M = \partial V/\partial t$ and $N = \partial V/\partial y$. Find V by integration:

$$V(t,\, y) = \int \frac{\partial V}{\partial t}\,dt = \int M\,dt = \int (4t + 4y + 3)\,dt = 2t^2 + 4ty + 3t + \varphi(y),$$

where $\varphi(y)$ is a function depending only on y. Now use

$$4t - 6y - 2 = N(t,\, y) = \frac{\partial V}{\partial y} = 4t + \frac{d}{dy}\varphi(y),$$

which implies that $\frac{d}{dy}\varphi(y) = -6y - 2$ so that $\varphi(y) = -3y^2 - 2y$. Hence, the solutions to the differential equation are given by the implicit solutions of $V(t,\, y) = C$, i.e.,

$$2t^2 + 4ty + 3t - 3y^2 - 2y = C,$$

where C is a constant. The initial condition $y(2) = 1$ implies that $C = 8+8+6-3-2 = 17$ so the solution to the initial value problem is given by the implicit equation

$$\boxed{2t^2 + 4ty + 3t - 3y^2 - 2y = 17.}$$

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5. **[12 Points]** Apply Picard's method to compute the approximations $y_1(t)$, $y_2(t)$, and $y_3(t)$ to the solution of the initial value problem

$$y' = 2y - t, \quad y(0) = 0.$$

▶ **Solution.** This has the standard form $y' = F(t, y)$ where $F(t, y) = 2y - t$, and Picard's approximation starts with the constant approximation $y_0(t) = y_0 = 0$. Then

$$\begin{aligned} y_1(t) &= y_0 + \int_0^t F(u, y_0(u))\, du = 0 + \int_0^t -u\, du = \boxed{-\frac{1}{2}t^2}; \\ y_2(t) &= \int_0^t F(u, y_1(u))\, du = \int_0^t (-u^2 - u)\, du = \boxed{-\frac{1}{3}t^3 - \frac{1}{2}t^2}; \\ y_3(t) &= \int_0^t F(u, y_2(u))\, du = \int_0^t (\frac{2}{3}u^3 - u^2 - u)\, du = \boxed{-\frac{1}{6}t^4 - \frac{1}{3}t^3 - \frac{1}{2}t^2}. \end{aligned}$$

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6. **[16 Points]** A 2000 gallon tank is initially full of brine which contains 100 pounds of salt. A solution containing 3.0 pounds of salt per gallon enters the tank at a flow rate of 4 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same flow rate of 4 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time t.

   (a) What is $y(0)$? That is, how much salt is in the tank at time $t = 0$? $\boxed{y(0) = 100 \text{ lbs}}$

   (b) Find the amount $y(t)$ of salt in the tank for all times t.

   ▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 3.0 lb/gal × 4 gal/min, i.e., 12 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 4.$$

Since mixture is entering and leaving at the same volume rate of 4 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 2000$. Hence $y(t)$ satisfies the equation

$$y' = 12 - \frac{4}{2000}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$y' + \frac{1}{500}y = 12, \qquad y(0) = 100.$$

This is a linear differential equation with integrating factor $\mu(t) = e^{t/500}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{t/500}y\right) = 12e^{t/500}.$$

Integration of this equation gives

$$e^{t/500}y = 6000e^{t/500} + C,$$

where C is an integration constant. Dividing by $e^{t/500}$ gives

$$y = 6000 + Ce^{-t/500},$$

and the initial condition $y(0) = 100$ gives a value of $C = -5900$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 6000 - 5900e^{-t/500}.}$$

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(c) How much salt is in the tank after 30 minutes?

▶ **Solution.** This is obtained by taking $t = 30$ (minutes) in the previous equation:

$$\boxed{y(30) = 6000 - 5900e^{-30/500} \approx 443.589 \text{ lb}}$$

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(d) What is $\lim_{t\to\infty} y(t)$? $\boxed{6000 \text{ lb}}$

Exam I Supplementary Sheet

Some Integral Formulas

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (if $n \neq -1$)
2. $\int \dfrac{1}{x}\,dx = \ln|x| + C$
3. $\int \dfrac{1}{a+bx}\,dx = \dfrac{1}{b}\ln|a+bx| + C \quad (b \neq 0)$
4. $\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C \quad (a > 0)$
5. $\int \dfrac{1}{x(a+bx)}\,dx = \dfrac{1}{a}\ln\left|\dfrac{x}{a+bx}\right| + C$
6. $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C$
7. $\int \ln x\,dx = x\ln x - x + C$
8. $\int xe^{ax}\,dx = \dfrac{xe^{ax}}{a} - \dfrac{e^{ax}}{a^2} + C$