Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolution products, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

1. [**14 Points**]

   (a) Complete the following definition: Suppose $f(t)$ is a continuous function defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows:

   $$F(s) = \mathcal{L}\{f(t)\}(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

   for all s sufficiently large.

   (b) Using your definition compute the Laplace transform of $f(t) = 3e^{-5t} + 2$.

   ▶ **Solution.**

   $$\begin{aligned} \mathcal{L}\left\{3e^{-5t} + 2\right\}(s) &= \int_0^\infty e^{-st}(3e^{-5t} + 2)\, dt \\ &= \int_0^\infty 3e^{-(s+5)t}\, dy + 2\int_0^\infty e^{-st}\, dt \\ &= \left.\frac{3e^{-(s+5)t}}{-(s+5)}\right|_0^\infty + 2\left.\frac{e^{-st}}{-s}\right|_0^\infty \\ &= \frac{3}{s+5} + \frac{2}{s}. \end{aligned}$$

   ◀

2. [**21 Points**] Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) in the table.

   (a) $f_1(t) = e^{-2t}(3t^4 + 5\cos 3t)$

   ▶ **Solution.** $f_1(t) = 3t^4e^{-2t} + 5e^{-2t}\cos 3t$ so $\boxed{F_1(s) = 3\dfrac{4!}{(s+2)^5} + 5\dfrac{s+2}{(s+2)^2+9}}$, where we have used formulas 1, 7, and 10 from the Laplace Transform Table. ◀

   (b) $f_2(t) = t \sin 2t$

   ▶ **Solution.** Since $\mathcal{L}\{\sin 2t\}(s) = \dfrac{2}{s^2+4}$ (formula 9), it follows from formula 3 that

   $$F_2(s) = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = 2(s^2+4)^{-2}2s = \boxed{\frac{4s}{(s^2+4)^2}}.$$

   ◀

(c) $f_3(t) = t^3 * \sin 2t$ (Recall that $f * g$ is the *convolution* product of f and g.

▶ **Solution.** From formulas 15, 5 and 9, we get

$$F_3(s) = \frac{3!}{s^4} \times \frac{2}{s^2+4} = \boxed{\frac{12}{s^4(s^2+4)}}.$$

◀

3. **[21 Points]** Compute the inverse Laplace transform of each of the following rational functions.

(a) $F(s) = \dfrac{2s^2 - 3s + 1}{s^3}$

▶ **Solution.** $F(s) = 2\dfrac{1}{s} - 3\dfrac{1}{s^2} + \dfrac{1}{s^3}$, so applying formulas 1 and 5 we get

$$\boxed{f(t) = 2 - 3t + \frac{1}{2}t^2}.$$

◀

(b) $G(s) = \dfrac{s^2+4}{(s+1)(s+2)(s-2)}$

▶ **Solution.** Using partial fractions, we conclude that

$$G(s) = \frac{s^2+4}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2},$$

where

$$\begin{aligned} A &= \left.\frac{s^2+4}{(s+2)(s-2)}\right|_{s=-1} = \frac{(-1)^2+4}{(1)(-3)} = -\frac{5}{3}, \\ B &= \left.\frac{s^2+4}{(s+1)(s-2)}\right|_{s=-2} = \frac{(-2)^2+4}{(-1)(-4)} = 2, \text{ and} \\ C &= \left.\frac{s^2+4}{(s+1)(s+2)}\right|_{s=2} = \frac{(2)^2+4}{3)(4)} = \frac{2}{3}. \end{aligned}$$

Thus, $G(s) = \frac{-5/3}{s+1} + \frac{2}{s+2} + \frac{2/3}{s-2}$, so that

$$\boxed{g(t) = -\frac{5}{3}e^{-t} + 2e^{-2t} + \frac{2}{3}e^{2t}.}$$

◀

(c) $H(s) = \dfrac{2s+5}{s^2+4s+13}$

▶ **Solution.** Rewriting $H(s)$ as

$$H(s) = \frac{2s+5}{s^2+4s+13} = \frac{2s+5}{(s+2)^2+9} = \frac{2s+4+1}{(s+2)^2+9} = \frac{2(s+2)}{(s+2)^2+9} + \frac{1}{(s+2)^2+9},$$

formulas 10 and 11 give

$$\boxed{h(t) = 3e^{-2t}\cos 3t + \frac{1}{3}e^{-2t}\sin 3t.}$$

◀

4. [**24 Points**] Find the *characteristic polynomial* and the general solution of each of the following constant coefficient linear homogeneous differential equations:

(a) $y'' - 4y' + 9y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 - 4s + 9 = (s-2)^2 + 5$, which has roots $2 \pm \sqrt{5}i$. Hence, the general solution is

$$\boxed{y = c_1e^{2t}\cos\sqrt{5}t + c_2e^{2t}\sin\sqrt{5}t.}$$

◀

(b) $y'' - 6y' + 9y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 - 6s + 9 = (s-3)^2$ which has a single root 3 of multiplicity 2. Thus, the general solution is

$$\boxed{y = c_2e^{3t} + c_2te^{3t}.}$$

◀

(c) $y'' - 10y' + 9y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 - 10s + 9 = (s-1)(s-9)$, which has roots 1 and 9. Thus, the general solution is

$$\boxed{y = c_1e^{t} + c_2e^{9t}.}$$

◀

(d) $y''' + 9y' = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^3 + 9s = s(s^2+9)$, which has roots 0 and $\pm 3i$. Thus, the general solution is

$$\boxed{c + 1 + c_2\cos 3t + c_3\sin 3t.}$$

◀

5. [**20 Points**] Use the Laplace transform method to find the solution of the following initial value problem:

$$y'' + 4y' + 4y = te^{-2t}, \qquad y(0) = 1,\ y'(0) = -2.$$

$\blacktriangleright$ **Solution.** Let $Y(s) = \mathcal{L}\left\{y(t)\right\}(s)$ where $y(t)$ is the unknown solution to the initial value problem. Then apply the Laplace transform to the differential equation and use the differentiation formulas 12 and 13 and the formula 7 to get an equation for $Y(s)$:

$$s^2Y - s + 2 + 4(sY - 1) + 4Y = \frac{1}{(s+2)^2}.$$

Collecting terms in this equation for Y gives

$$(s^2 + 4s + 4)Y = s + 2 + \frac{1}{(s+2)^2},$$

and solving for Y gives

$$Y = \frac{s+2}{(s+2)^2} + \frac{1}{(s+2)^4} = \frac{1}{s+2} + \frac{(s+2)^4}{.}$$

Applying the inverse Laplace transform to Y then gives:

$$\boxed{y(t) = e^{-2t} + \frac{1}{6}t^3e^{-2t}.}$$

$\blacktriangleleft$

Exam II Supplementary Sheets

A Short Table of Laplace Transforms

1.	$\mathcal{L}\left\{af(t)+bg(t)\right\}(s)$	$=$	$aF(s)+bG(s)$
2.	$\mathcal{L}\left\{e^{at}f(t)\right\}(s)$	$=$	$F(s-a)$
3.	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
4.	$\mathcal{L}\left\{1\right\}(s)$	$=$	$\dfrac{1}{s}$
5.	$\mathcal{L}\left\{t^n\right\}(s)$	$=$	$\dfrac{n!}{s^{n+1}}$
6.	$\mathcal{L}\left\{e^{at}\right\}(s)$	$=$	$\dfrac{1}{s-a}$
7.	$\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s)$	$=$	$\dfrac{n!}{(s-\alpha)^{n+1}}$
8.	$\mathcal{L}\left\{\cos bt\right\}(s)$	$=$	$\dfrac{s}{s^2+b^2}$
9.	$\mathcal{L}\left\{\sin bt\right\}(s)$	$=$	$\dfrac{b}{s^2+b^2}$
10.	$\mathcal{L}\left\{e^{at}\cos bt\right\}(s)$	$=$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$\mathcal{L}\left\{e^{at}\sin bt\right\}(s)$	$=$	$\dfrac{b}{(s-a)^2+b^2}$
12.	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$sF(s)-f(0)$
13.	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2F(s)-sf(0)-f'(0)$
14.	$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\}(s)$	$=$	$\dfrac{F(s)}{s}$
15.	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$

Table of Convolutions

	$f(t)$	$g(t)$	$f * g(t)$
1.	t	t^n	$\dfrac{t^{n+2}}{(n+1)(n+2)}$
2.	t	$\sin at$	$\dfrac{at - \sin at}{a^2}$
3.	t^2	$\sin at$	$\dfrac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
4.	t	$\cos at$	$\dfrac{1 - \cos at}{a^2}$
5.	t^2	$\cos at$	$\dfrac{2}{a^3}(at - \sin at)$
6.	t	e^{at}	$\dfrac{e^{at} - (1 + at)}{a^2}$
7.	t^2	e^{at}	$\dfrac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
8.	e^{at}	e^{bt}	$\dfrac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
9.	e^{at}	e^{at}	te^{at}
10.	e^{at}	$\sin bt$	$\dfrac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
11.	e^{at}	$\cos bt$	$\dfrac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
12.	$\sin at$	$\sin bt$	$\dfrac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
13.	$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at - at\cos at)$
14.	$\sin at$	$\cos bt$	$\dfrac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
15.	$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$
16.	$\cos at$	$\cos bt$	$\dfrac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
17.	$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=\lambda} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1 s + C_1\big|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$