Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned}\sin(\theta+\varphi) &= \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi\end{aligned}}$$

1. Solve: **[12 Points]** $t^2y'' + 7ty' + 9y = 0$.

   ▶ **Solution.** The indicial equation is $q(s) = s(s-1) + 7s + 9 = s^2 + 6s + 9 = (s+3)^2$ so there is a single root -3 of multiplicity 2. Hence the solution is

$$\boxed{y = c_1t^{-3} + c_2t^{-3}\ln t.}$$

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2. Solve: **[16 Points]** $y'' + 4y' + 3y = 9t$.

   ▶ **Solution.** We will use the method of incomplete partial fractions from Section 4.5. Start by looking for a solution y subject to the initial conditions $y(0) = y'(0) = 0$. Applying the Laplace transformation to the differential equation, where $Y = \mathcal{L}\{y(t)\}$ gives:

$$s^2Y + 4sY + 3Y = \frac{9}{s^2}.$$

   Solving for Y gives:

$$Y = \frac{9}{s^2(s^2+4s+3)} = \frac{9}{s^2(s+3)(s+1)}.$$

   From the partial fraction theorem we can write:

$$Y = \frac{A}{s^2} + \frac{p_1(s)}{s(s+1)(s+3)}$$

   where $A = \left.\dfrac{9}{(s+1)(s+3)}\right|_{s=0} = \dfrac{9}{3} = 3$, and

$$p_1(s) = \frac{9 - 3(s+1)(s+3)}{s} = \frac{-3s^2 - 12s}{s} = -3s - 12.$$

   Continuing,

$$\frac{-3s-12}{s(s+1)(s+3)} = \frac{B}{s} + \frac{p_2(s)}{(s+1)(s+3)}$$

where $B = \left.\dfrac{-3s-12}{(s+1)(s+3)}\right|_{s=0} = \dfrac{-12}{3} = -4$. Since, $p(s) = (s+1)(s+3) = s^2+4s+3$ is the characteristic polynomial of the associated homogeneous equation, it follows that $\mathcal{L}^{-1}\left\{\dfrac{p_2(s)}{(s+3)(s+1)}\right\}$ is a particular solution of the homogeneous equation

$$y'' + 4y' + 3y = 0$$

and hence can be included in the homogeneous part of the solution y_h. Thus, for y_p we get

$$y_p(t) = \mathcal{L}^{-1}\left\{\frac{3}{s^2} + \frac{-4}{s}\right\} = 3t - 4.$$

Then, $y_g = y_p + y_h$ so

$$\boxed{y_g(t) = 3t - 4 + c_1e^{-t} + c_2e^{-3t}.}$$

Alternate Solution: One may also use the alternative method of undetermined coefficients from Section 4.4. For this method, the characteristic polynomial of the associated homogeneous equation $y'' + 4y' + 3y = 0$ is $q(s) = s^2 + 4s + 3 = (s+3)(s+1)$ and the Laplace transform of the right hand side is $\mathcal{L}\{9t\} = \dfrac{9}{s^2}$, which has denominator $v(s) = s^2$. Thus, the polynomial $q(s)v(s) = s^2(s+3)(s+1)$ and the corresponding basis for the homogeneous equation with polynomial $q(s)v(s)$ is $\mathcal{B}_{q(s)v(s)} = \{e^{-t}, e^{-3t}, 1, t\}$ and the basis $\mathcal{B}_{q(s)} = \{e^{-t}, e^{-3t}\}$. Thus $\mathcal{B}_{q(s)v(s)} \setminus \mathcal{B}_{q(s)} = \{1, t\}$. It follows that a particular solution $y_p = A + Bt$ for some constants A and B to be determined by substitution into the original equation. Computing $y_p' = B$ and $y_p'' = 0$ it follows that

$$y_p'' + 4y_p' + 3y_p = 0 + 4B + 3(A + Bt) = 3Bt + (4B + 3A) = 9t.$$

Hence, by equating the constant terms and the coefficients of t on the two sides of the last equality, we get $3B = 9$ and $4B + 3A = 0$. Hence, $B = 3$ and substituting in the second equation gives $A = -4$. Thus $y_p = 3t - 4$, and as with the first method, $y_g = y_p + y_h$ so

$$\boxed{y_g(t) = 3t - 4 + c_1e^{-t} + c_2e^{-3t}.}$$

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3. [**20 Points**] You may assume that $\mathcal{S} = \{e^{-2t}, te^{-2t}\}$ is a fundamental set of solutions for the homogeneous equation

$$y'' + 4y' + 4y = 0.$$

Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

$$y'' + 4y' + 4y = t^{-3}e^{-2t}.$$

▶ **Solution.** The particular solution $y_p(t)$ has the form

$$y_p = u_1y_1 + u_2y_2 = u_1e^{-2t} + u_2te^{-2t}$$

where u_1 and u_2 are unknown functions whose derivatives satisfy the following equations:

$$\begin{aligned} u_1'e^{-2t} + u_2'te^{-2t} &= 0 \\ u_1'(-2e^{-2t}) + u_2'(e^{-2t} - 2te^{-2t}) &= t^{-3}e^{-2t}. \end{aligned}$$

Adding 2 times the first equation to the second shows that $u_2' = t^{-3}$, and the first equation then gives

$$u_1' = -tu_2' = -t \cdot t^{-3} = -t^{-2}.$$

Hence

$$\begin{aligned} u_1 &= \int u_1'\,dt = -\int t^{-2}\,dt = t^{-1} \\ u_2 &= \int u_2'\,dt = \int t^{-3}\,dt = -\frac{1}{2}t^{-2}, \end{aligned}$$

so that

$$y_p = t^{-1}e^{-2t} - \frac{1}{2}t^{-2}te^{-2t} = \frac{1}{2}t^{-1}e^{-2t}.$$

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4. [**14 Points**] Find the Laplace transform of the following function:

$$f(t) = \begin{cases} t^2 - 4t & \text{if } 0 \le t < 4, \\ 0 & \text{if } t \ge 4. \end{cases}$$

▶ **Solution.** First write $f(t)$ in terms of characteristic functions $\chi_{[a,b)}(t)$ and unit step functions $h(t-c)$:

$$\begin{aligned} f(t) &= (t^2 - 4t)\chi_{[0,4)}(t) + (0)\chi_{[2,\infty)}(t) \\ &= (t^2 - 4t)(h(t) - h(t-4)) \\ &= (t^2 - 4t) - (t^2 - 4t)h(t-4). \end{aligned}$$

Now apply the second translation formula (number 3 or 3' on the table):

$$\begin{aligned} F(s) &= \frac{2}{s^3} - \frac{4}{s} - e^{-4s}\mathcal{L}\left\{(t+4)^2 - 4(t+4)\right\} \\ &= \frac{2}{s^3} - \frac{4}{s} - e^{-4s}\mathcal{L}\left\{(t^2 + 8t + 16 - 4t - 16\right\} \\ &= \frac{2}{s^3} - \frac{4}{s} - e^{-4s}\mathcal{L}\left\{(t^2 + 4t\right\} \\ &= \boxed{\frac{2}{s^3} - \frac{4}{s} - e^{-4s}\left(\frac{2}{s^3} + \frac{4}{s}\right)}. \end{aligned}$$

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5. [**16 Points**] Find the inverse Laplace transform of the following functions:

   (a) $F(s) = \dfrac{1}{(s+1)^3}e^{-2s} + \dfrac{2}{s^4}e^{-4s}$

   ▶ **Solution.** Letting $F_1(s) = \dfrac{1}{(s+1)^3}$ we find that

$$f_1(t) = \mathcal{L}^{-1}\left\{F_1(s)\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^3}\right\} = \frac{1}{2}t^2e^{-t},$$

   and letting $F_2(s) = \dfrac{2}{s^4}$ we find that $f_2(t) = \mathcal{L}^{-1}\left\{\dfrac{2}{s^4}\right\} = \dfrac{1}{3}t^3$. Then the second translation formula, Formula 3, applied twice, gives:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{F_1(s)e^{-2s}\right\} + \mathcal{L}^{-1}\left\{F_2(s)e^{-4s}\right\} \\ &= f_1(t-2)h(t-2) + f_2(t-4)h(t-4) \\ &= \frac{1}{2}(t-2)^2e^{-(t-2)}h(t-2) + \frac{1}{3}(t-4)^3h(t-4). \end{aligned}$$

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   (b) $G(s) = \dfrac{2}{s^2+9}e^{-2\pi s}$

   ▶ **Solution.** Letting $G_1(s) = \dfrac{2}{s^2+9}$ we find that

$$g_1(t) = \mathcal{L}^{-1}\left\{G_1(s)\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s^2+9}\right\} = \frac{2}{3}\sin 3t.$$

   Then the second translation formula, Formula 3, gives:

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = \mathcal{L}^{-1}\left\{G_1(s)e^{-2\pi s}\right\} = g_1(t-2\pi)h(t-2\pi) = (\frac{2}{3}\sin 3(t-2\pi))h(t-2\pi).$$

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6. [**22 Points**] Solve the following initial value problem:

$$y'' + 16y = h(t-\pi) - h(t-3\pi), \qquad y(0) = 1,\ y'(0) = 0.$$

▶ **Solution.** Let $Y(s) = \mathcal{L}\left\{y(t)\right\}$ be the Laplace transform of the solution function. Applying the Laplace transform to the differential equation gives

$$s^2Y(s) - s + 16Y(s) = \mathcal{L}\left\{h(t-\pi) - h(t-3\pi)\right\} = \frac{e^{\pi s}s}{-}\frac{e^{-3\pi s}}{s}.$$

Thus,

$$Y(s) = \frac{s}{s^2+16} + e^{-\pi s}\left(\frac{1}{s(s^2+16)}\right) - e^{-3\pi s}\left(\frac{1}{s(s^2+16)}\right).$$

Using partial fractions, we find that

$$\frac{1}{s(s^2+16)} = \frac{1}{16}\left(\frac{1}{s} - \frac{2}{s^2+16}\right).$$

Applying the second translation formula (Formula 3) twice gives:

$$y(t) = \cos 4t + \left(\frac{1}{16} - \frac{1}{16}\cos 4(t-\pi)\right)h(t-\pi) - \left(\frac{1}{16} - \frac{1}{16}\cos 4(t-3\pi)\right)h(t-3\pi).$$

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Exam III Supplementary Sheets

A Short Table of Laplace Transforms

1.	$\mathcal{L}\left\{af(t)+bg(t)\right\}(s)$	$=$	$aF(s)+bG(s)$
2.	$\mathcal{L}\left\{e^{at}f(t)\right\}(s)$	$=$	$F(s-a)$
3.	$\mathcal{L}\left\{f(t-c)h(t-c)\right\}$	$=$	$e^{-sc}F(s)$
3'.	$\mathcal{L}\left\{g(t)h(t-c)\right\}$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$
4.	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\frac{d}{ds}F(s)$
5.	$\mathcal{L}\left\{1\right\}(s)$	$=$	$\frac{1}{s}$
6.	$\mathcal{L}\left\{t^n\right\}(s)$	$=$	$\frac{n!}{s^{n+1}}$
7.	$\mathcal{L}\left\{e^{at}\right\}(s)$	$=$	$\frac{1}{s-a}$
8.	$\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s)$	$=$	$\frac{n!}{(s-\alpha)^{n+1}}$
9.	$\mathcal{L}\left\{\cos bt\right\}(s)$	$=$	$\frac{s}{s^2+b^2}$
10.	$\mathcal{L}\left\{\sin bt\right\}(s)$	$=$	$\frac{b}{s^2+b^2}$
11.	$\mathcal{L}\left\{e^{at}\cos bt\right\}(s)$	$=$	$\frac{s-a}{(s-a)^2+b^2}$
12.	$\mathcal{L}\left\{e^{at}\sin bt\right\}(s)$	$=$	$\frac{b}{(s-a)^2+b^2}$
13.	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$sF(s)-f(0)$
14.	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2F(s)-sf(0)-f'(0)$
15.	$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\}(s)$	$=$	$\frac{F(s)}{s}$
16.	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=\lambda} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1 s + C_1\big|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$