Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolution products, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

In Exercises 1 – 7, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' + 2y = 3e^{-2t} - 2e^{3t}, \quad y(0) = 2.$

▶ **Solution.** This is a first order linear equation, so compute an integrating factor

$$\mu(t) = e^{\int 2\,dt} = e^{2t},$$

and multiply the equation by $\mu(t) = e^{2t}$ to get $e^{2t}y' + 2e^{2t}y = 3 - 2e^{5t}$. The left hand side is $(e^{2t}y)'$ so we get the equation $(e^{2t}y)' = 3 - 2e^{5t}$ and integration then gives $e^{2t}y = 3t - \frac{2}{5}e^{5t} + C$ so that $y = 3te^{-2t} - \frac{2}{5}e^{3t} + Ce^{-2t}$. Using the initial condition $y(0) = 2$ gives $2 = y(0) = -\frac{2}{5} + C$, so that $C = 2 + \frac{2}{5} = \frac{12}{5}$. Hence,

$$\boxed{y = \left(3t + \frac{12}{5}\right)e^{-2t} - \frac{2}{5}e^{3t}.}$$

◀

2. [**12 Points**] $y' = y - y^2, \quad y(0) = 2.$

▶ **Solution.** This equation is separable, and separating the variables gives $\dfrac{y'}{y - y^2} = 1$, which we write in differential form as $\dfrac{dy}{y - y^2} = dt$. To integrate the left hand side, write it in partial fraction form:

$$\frac{1}{y - y^2} = \frac{1}{y(1 - y)} = \frac{-1}{y(y - 1)} = \frac{1}{y} - \frac{1}{y - 1}.$$

Integrating then gives $\ln y - \ln(y - 1) = t + C$, so that $\ln \dfrac{y}{y - 1} = t + C$. Taking the exponential of both sides gives $\dfrac{y}{y - 1} = Ke^t$ where K is a constant. Using the initial condition $y(0) = 2$ we conclude that $K = 2/(2 - 1) = 2$. Now solving for y gives $y = 2e^t(y - 1) = 2e^t y - 2e^t$. Hence $y(1 - 2e^t) = -2e^t$. Thus,

$$\boxed{y = \frac{-2e^t}{1 - 2e^t} = \frac{2e^t}{2e^t - 1}.}$$

Note that this equation is also a Bernoulli equation and can thus be solved by the substitution $z = y^{-1}$. ◀

3. [**12 Points**] $2y'' + 5y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 1.$

▶ **Solution.** This equation has characteristic polynomial

$$q(s) = 2s^2 + 5s + 2 = (2s+1)(s+2)$$

which has roots $-1/2$ and -2. Hence the general solution of the homogeneous equation is $y = c_1e^{-t/2} + c_2e^{-2t}$. The initial conditions mean that c_1 and c_2 satisfy

$$\begin{aligned} c_1 + c_2 &= 1 \\ (-1/2)c_1 - 2c_2 &= 1. \end{aligned}$$

Solve these equations to get $c_1 = 2$, $c_2 = -1$. Hence,

$$\boxed{y = 2e^{-t/2} - e^{-2t}.}$$

◀

4. [**12 Points**] $4t^2y'' + y = 0.$

▶ **Solution.** This is a Cauchy-Euler equation with indicial equation

$$q(s) = 4s(s-1) + 1 = 4s^2 - 4s + 1 = (2s-1)^2$$

so there is a single root $1/2$ with multiplicity 2. Hence,

$$\boxed{y = c_1t^{1/2} + c_2t^{1/2}\ln t.}$$

◀

5. [**12 Points**] $4y'' + 9y = 0, \quad y(0) = -1,\ y'(0) = 6.$

▶ **Solution.** This is a constant coefficient homogeneous linear equation with characteristic polynomial $q(s) = 4s^2 + 9$ which has roots $\pm\frac{3}{2}i$. Hence, the general solution is $y = c_1\cos(3/2)t + c_2\sin(3/2)t$. The initial conditions imply that $-1 = y(0) = c_1$ and $6 = y'(0) = (3/2)c_2$ so that $c_2 = 4$. Thus

$$\boxed{y = -\cos\frac{3t}{2} + 4\sin\frac{3t}{2}.}$$

◀

6. [**12 Points**] $y'' + 6y' + 9y = 12e^{-3t}.$

▶ **Solution.** The method of undetermined coefficients applies to this equation. We will use undetermined coefficients in the form of incomplete partial fractions. The characteristic polynomial is $q(s) = s^2 + 6s + 9 = (s+3)^2$, so the solution of the homogeneous equation is

$$y_y = c_1 e^{-3t} + c_2 t e^{-3t}.$$

The Laplace transform of the right hand side is $\mathcal{L}\{12e^{-3t}\} = \dfrac{12}{s+3}$. Let $y(t)$ be the solution of the differential equation with initial values $y(0) = y'(0) = 0$, so that if $Y(s) = \mathcal{L}\{y(t)\}$, we get that $Y(s)$ satisfies the equation

$$s^2Y + 6sY + 9Y = \frac{12}{s+3}.$$

Solving for Y gives

$$Y = \frac{12}{(s+3)^3}.$$

It follows that a particular solution of the nonhomogeneous equation is

$$y_p(t) = \mathcal{L}^{-1}\left\{\frac{12}{(s+3)^3}\right\} = 6t^2e^{-3t}.$$

Hence the general solution is

$$\boxed{y = 6t^2e^{-3t} + c_1e^{-3t} + c_2te^{-3t}.}$$

◀

7. [**12 Points**] $y'' + y = 3h(t-\pi)$, $\quad y(0) = 1$, $y'(0) = -3$. Recall that $h(t)$ refers to the unit step function.

▶ **Solution.** Use the Laplace transform method. Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the unknown solution of the initial value problem. Applying the Laplace transform to the differential equation gives:

$$s^2Y - s + 3 + Y = 3\frac{e^{-\pi s}}{s}.$$

Solve for Y:

$$\begin{aligned} Y &= \frac{s-3}{s^2+1} + \frac{3}{s(s^2+1)}e^{-\pi s} \\ &= \frac{s}{s^2+1} - \frac{3}{s^2+1} + \frac{3}{s}e^{-\pi s} - \frac{3s}{s^2+1}e^{-\pi s}, \end{aligned}$$

where the last equation is a consequence of the partial fraction decomposition:

$$\frac{3}{s(s^2+1)} = \frac{3}{s} - \frac{3s}{s^2+1}.$$

Then take the inverse laplace transform to get:

$$\boxed{y = \cos t - 3\sin t + 3h(t-\pi) - 3h(t-\pi)\cos(t-\pi).}$$

◀

8. [**12 Points**] Find a particular solution of the differential equation

$$t^2y'' + ty' - 4y = t^3,$$

given the fact that the general solution of the associated homogeneous equation is

$$y_h = c_1t^2 + c_2t^{-2}.$$

▶ **Solution.** Use variation of parameters. A particular solution is given by

$$y_p = u_1t^2 + u_2t^{-2}$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1't^2 + u_2't^{-2} &= 0 \\ 2u_1't - 2u_2't^{-3} &= t. \end{aligned}$$

Multiplying the first equation by $2t^{-1}$ and adding to the second eliminates u_2', giving $4u_1't = t$, or $u_1' = (1/4)$. Substituting in the first equation then gives

$$u_2' = -u_1't^4 = -\frac{1}{4}t^4.$$

Integrating then gives

$$u_1 = \frac{t}{4} \quad \text{and} \quad u_2 = -\frac{1}{20}t^5,$$

which gives

$$y_p = \frac{1}{4}tt^2 - \frac{1}{20}t^{-2}t^5 = \frac{1}{5}t^3.$$

◀

9. [**12 Points**] Find the Laplace transform of each of the following functions.

(a) $f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2, \\ 8-2t & \text{if } 2 \le t < 4 \\ 0 & \text{if } t \ge 4. \end{cases}$

▶ **Solution.** First write $f(t)$ in terms of the heaviside functions:

$$\begin{aligned} f(t) &= t^2\chi_{[0,\,2)} + (8-2t)\chi_{[2,\,4)} \\ &= t^2(h(t) - h(t-2)) + (8-2t)(h(t-2) - h(t-4)) \\ &= t^2 + (8-2t-t^2)h(t-2) + (2t-8)h(t-4). \end{aligned}$$

Then

$$\begin{aligned} F(s) &= \frac{2}{s^3} + e^{-2s}\mathcal{L}\left\{8 - 2(t+2) - (t+2)^2\right\} + e^{-4s}\mathcal{L}\left\{2(t+4) - 8\right\} \\ &= \frac{2}{s^3} + e^{-2s}\mathcal{L}\left\{8 - 2t - 4 - t^2 - 4t - 4\right\} + e^{-4s}\mathcal{L}\left\{2t\right\} \\ &= \frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{6}{s^2}\right) + e^{-4s}\left(\frac{2}{s^2}\right). \end{aligned}$$

◀

(b) $g(t) = te^{2t}\cos 3t$

▶ **Solution.** $\mathcal{L}\{e^{2t}\cos 3t\} = \dfrac{s-2}{(s-2)^2+9}$ so that

$$\begin{aligned}
G(s) &= \mathcal{L}\left\{te^{2t}\cos 3t\right\} = -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+9}\right)\\
&= -\frac{((s-2)^2+9)-(s-2)2(s-2)}{((s-2)^2+9)^2}\\
&= \frac{(s-2)^2-9}{((s-2)^2+9)^2}.
\end{aligned}$$

◀

10. [**12 Points**] Compute each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{2s+5}{s^2+4s+13}\right\}$

▶ **Solution.**

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+4s+13}\right\} &= \mathcal{L}^{-1}\left\{\frac{2s+5}{(s+2)^2+9}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{2(s+2)}{(s+2)^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2+9}\right\}\\
&= \boxed{2e^{-2t}\cos 3t + \frac{1}{3}e^{-2t}\sin 3t}.
\end{aligned}$$

◀

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{(s+1)(s^2-4)}\right\}$

▶ **Solution.** First use partial fractions to get

$$\frac{4s}{(s+1)(s^2-4)} = \frac{4s}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2},$$

where

$$A = \left.\frac{4s}{s^2-4}\right|_{s=-1} = \frac{4}{3}, \qquad B = \left.\frac{4s}{(s+1)(s-2)}\right|_{s=-2} = -2,$$

and

$$C = \left.\frac{4s}{(s+1)(s+2)}\right|_{s=2} = \frac{2}{3}.$$

This gives

$$\mathcal{L}^{-1}\left\{\frac{4s}{(s+1)(s^2-4)}\right\} = \boxed{\frac{4}{3}e^{-t} - 2e^{-2t} + \frac{2}{3}e^{2t}.}$$

◀

11. [**18 Points**] Let $A = \begin{bmatrix} -7 & 8 \\ -2 & 1 \end{bmatrix}$.

(a) Compute $(sI - A)^{-1}$.

▶ **Solution.** $(sI - A) = \begin{bmatrix} s+7 & -8 \\ 2 & s-1 \end{bmatrix}$ so

$$p(s) = \det(sI - A) = (s+7)(s-3) + 16 = s^2 + 6s + 9 = (s+3)^2.$$

Hence

$$(sI - A)^{-1} = \begin{bmatrix} \dfrac{s-1}{(s+3)^2} & \dfrac{8}{(s+3)^2} \\ \dfrac{-2}{(s+3)^2} & \dfrac{s+7}{(s+3)^2} \end{bmatrix}.$$

◀

(b) Find $\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

▶ **Solution.**

$$\begin{aligned} \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} &= \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{s-1}{(s+3)^2} & \dfrac{8}{(s+3)^2} \\ \dfrac{-2}{(s+3)^2} & \dfrac{s+7}{(s+3)^2} \end{bmatrix}\right\} \\ &= \mathcal{L}^{-1}\left\{\begin{bmatrix} \dfrac{(s+3)-4}{(s+3)^2} & \dfrac{8}{(s+3)^2} \\ \dfrac{-2}{(s+3)^2} & \dfrac{(s+3)+4}{(s+3)^2} \end{bmatrix}\right\} \\ &= \begin{bmatrix} e^{-3t} - 4te^{-3t} & 8te^{-3t} \\ -2te^{-3t} & e^{-3t} + 4te^{-3t} \end{bmatrix} \end{aligned}$$

◀

(c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.**

$$\begin{aligned} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} e^{-3t} - 4te^{-3t} & 8te^{-3t} \\ -2te^{-3t} & e^{-3t} + 4te^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} c_1e^{-3t} + (8c_2 - 4c_1)te^{-3t} \\ c_2e^{-3t} + (4c_2 - 2c_1)te^{-3t} \end{bmatrix}. \end{aligned}$$

◀

(d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

▶ **Solution.** Just set $c_1 = 1$ and $c_2 = 2$ in Part (c):

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{-3t} + 12te^{-3t} \\ 2e^{-3t} + 6te^{-3t} \end{bmatrix}.$$

◀

12. [**12 Points**] A tank initially contains 500 gallons of water in which 20 pounds of salt is initially dissolved in the water. Brine (a water-salt mixture) containing 0.5 pounds of salt per gallon flows into the tank at the rate of 4 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the same rate of 4 gal/min.

(a) Find the amount of salt $y(t)$ in the tank at time t.

▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0.5 lb/gal $\times$ 4 gal/min, i.e., 2 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 4.$$

Since mixture is entering and leaving at the same volume rate of 4 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 500$. Hence $y(t)$ satisfies the equation

$$y' = 2 - \frac{4}{500}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$y' + \frac{1}{125}y = 2, \qquad y(0) = 20.$$

This is a linear differential equation with integrating factor $\mu(t) = e^{t/125}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{t/125}y\right) = 2e^{t/125}.$$

Integration of this equation gives

$$e^{t/125}y = 250e^{t/125} + C,$$

where C is an integration constant. Dividing by $e^{t/125}$ gives

$$y = 250 + Ce^{-t/125},$$

and the initial condition $y(0) = 20$ gives a value of $C = -230$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 250 - 230e^{-t/125}.}$$

◀

(b) How much salt does the tank contain after 1 hour?

▶ **Solution.** This is obtained by taking $t = 60$ (minutes) in the previous equation:

$$\boxed{y(60) = 250 - 230e^{-60/125} \approx 107.68 \text{ lb}}$$

◀

(c) What is $\lim_{t\to\infty} y(t)$?

▶ **Solution.** $\boxed{250 \text{ lb}}$ ◀

A Short Table of Laplace Transforms

1.	$\mathcal{L}\left\{af(t)+bg(t)\right\}(s)$	$=$	$aF(s)+bG(s)$
2.	$\mathcal{L}\left\{e^{at}f(t)\right\}(s)$	$=$	$F(s-a)$
3.	$\mathcal{L}\left\{f(t-c)h(t-c)\right\}$	$=$	$e^{-sc}F(s)$
3'.	$\mathcal{L}\left\{g(t)h(t-c)\right\}$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$
4.	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\frac{d}{ds}F(s)$
5.	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$sF(s)-f(0)$
6.	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2F(s)-sf(0)-f'(0)$
7.	$\mathcal{L}\left\{\int_0^t f(x)\,dx\right\}(s)$	$=$	$\frac{F(s)}{s}$
8.	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$
9.	$\mathcal{L}\left\{1\right\}(s)$	$=$	$\frac{1}{s}$
10.	$\mathcal{L}\left\{t^n\right\}(s)$	$=$	$\frac{n!}{s^{n+1}}$
11.	$\mathcal{L}\left\{e^{at}\right\}(s)$	$=$	$\frac{1}{s-a}$
12.	$\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s)$	$=$	$\frac{n!}{(s-\alpha)^{n+1}}$
3.	$\mathcal{L}\left\{\cos bt\right\}(s)$	$=$	$\frac{s}{s^2+b^2}$
14.	$\mathcal{L}\left\{\sin bt\right\}(s)$	$=$	$\frac{b}{s^2+b^2}$
15.	$\mathcal{L}\left\{e^{at}\cos bt\right\}(s)$	$=$	$\frac{s-a}{(s-a)^2+b^2}$
16.	$\mathcal{L}\left\{e^{at}\sin bt\right\}(s)$	$=$	$\frac{b}{(s-a)^2+b^2}$
17.	$\mathcal{L}\left\{h(t-c)\right\}(s)$	$=$	$\frac{e^{-sc}}{s}$

Table of Convolutions

	$f(t)$	$g(t)$	$f * g(t)$
1.	t	t^n	$\dfrac{t^{n+2}}{(n+1)(n+2)}$
2.	t	$\sin at$	$\dfrac{at - \sin at}{a^2}$
3.	t^2	$\sin at$	$\dfrac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
4.	t	$\cos at$	$\dfrac{1 - \cos at}{a^2}$
5.	t^2	$\cos at$	$\dfrac{2}{a^3}(at - \sin at)$
6.	t	e^{at}	$\dfrac{e^{at} - (1 + at)}{a^2}$
7.	t^2	e^{at}	$\dfrac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
8.	e^{at}	e^{bt}	$\dfrac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
9.	e^{at}	e^{at}	te^{at}
10.	e^{at}	$\sin bt$	$\dfrac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
11.	e^{at}	$\cos bt$	$\dfrac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
12.	$\sin at$	$\sin bt$	$\dfrac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
13.	$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at - at\cos at)$
14.	$\sin at$	$\cos bt$	$\dfrac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
15.	$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$
16.	$\cos at$	$\cos bt$	$\dfrac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
17.	$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=\lambda} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$\left. B_1 s + C_1 \right|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$