Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned} \sin(\theta+\varphi) &= \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi \end{aligned}}$$

1. [**20 Points**] Find the general solution of the following Cauchy-Euler equations:

   (a) $3t^2y'' - 7ty' + 3y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = 3s(s-1) - 7s + 3 = 3s^2 - 10s + 3 = (3s-1)(s-3),$$

   which has the two distinct real roots $1/3$ and 3. Hence the general solution is

   $$\boxed{y = c_1t^{1/3} + c_2t^3.}$$

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   (b) $t^2y'' - ty' + 10y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = s(s-1) - s + 1 = s^2 - 2s + 10 = (s-1)^2 + 9,$$

   which has the complex roots $1 \pm 3i$. Hence the general solution is

   $$\boxed{y = c_1t\cos(3\ln|t|) + c_2t\sin(3\ln|t|).}$$

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2. [**20 Points**] Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

   $$y'' - 4y' + 4y = t^{1/2}e^{2t}.$$

   You may assume that the solution of the homogeneous equation $y'' - 4y' + 4y = 0$ is $y_h = c_1e^{2t} + c_2te^{2t}$.

   ▶ **Solution.** Letting $y_1 = e^{2t}$ and $y_2 = te^{2t}$, a particular solution has the form

   $$y_p = u_1y_1 + u_2y_2 = u_1e^{2t} + u_2te^{2t},$$

where u_1 and u_2 are unknown functions whose derivatives satisfy the simultaneous equations

$$\begin{aligned} u_1' e^{2t} + u_2' t e^{2t} &= 0 \\ u_1'(2e^{2t} + u_2'(e^{2t} + 2te^{2t}) &= t^{1/2} e^{2t}. \end{aligned}$$

Dividing both equations by e^{2t} gives the simpler equations

$$\begin{aligned} u_1' + u_2' t &= 0 \\ 2u_1' + u_2'(1+2t) &= t^{1/2}. \end{aligned}$$

Applying Cramer's rule gives

$$u_1' = \frac{\begin{vmatrix} 0 & t \\ t^{1/2} & 1+2t \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 1+2t \end{vmatrix}} = \frac{-t^{3/2}}{1} = -t^{3/2}$$

and

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 2 & t^{1/2} \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 2 & 1+2t \end{vmatrix}} = \frac{t^{1/2}}{1} = t^{1/2}$$

Integrating gives $u_1 = -\frac{2}{5}t^{5/2}$ and $u_2 = \frac{2}{3}t^{3/2}$ so that

$$\boxed{y_p = -\frac{2}{5}t^{5/2}e^{2t} + \frac{2}{3}t^{3/2}te^{2t} = \frac{4}{15}t^{5/2}e^{2t}.}$$

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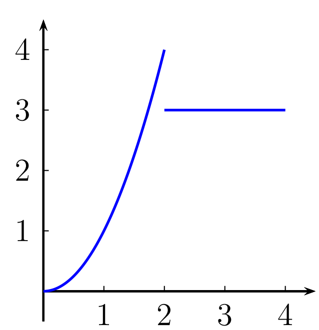
3. [**20 Points**] Let f be the function defined by

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 2, \\ 3 & \text{if } t \ge 2. \end{cases}$$

(a) Sketch the graph of $f(t)$ over the interval [0, 4].



(b) Find the Laplace transform of $f(t)$.

▶ **Solution.** Use characteristic functions to write $f(t)$ in terms of unit step functions:

$$\begin{aligned} f(t) &= t^2\chi_{[0,\,2)}(t) + 3\chi_{[2,\,\infty)}(t) \\ &= t^2(h(t) - h(t-2)) + 3h(t-2) \\ &= t^2 + (3 - t^2)h(t-2). \end{aligned}$$

Now apply the second translation theorem to get

$$\begin{aligned} F(s) = \mathcal{L}\left\{f(t)\right\} &= \frac{2}{s^3} + e^{-2s}\mathcal{L}\left\{3 - (t+2)^2\right\} \\ &= \frac{2}{s^3} + e^{-2s}\mathcal{L}\left\{3 - (t^2 + 4t + 4)\right\} \\ &= \frac{2}{s^3} + e^{-2s}\mathcal{L}\left\{-t^2 - 4t - 1)\right\} \\ &= \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{1}{s}\right). \end{aligned}$$

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4. **[20 Points]** Find the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{e^{-4s}}{(s+2)^3}$

▶ **Solution.** $F(s) = F_1(s)e^{-4s}$ where $F_1(s) = \dfrac{1}{(s+2)^3}$. Let

$$f_1(t) = \mathcal{L}^{-1}\left\{F_1(s)\right\} = \frac{1}{2}t^2e^{-2t}.$$

Then the inverse of the second translation theorem gives

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} &= f_1(t-4)h(t-4) \\ &= \frac{1}{2}(t-4)^2e^{-2(t-4)}h(t-4) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 4 \\ \frac{1}{2}(t-4)^2e^{-2(t-4)} & \text{if } t \ge 4. \end{cases} \end{aligned}$$

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(b) $G(s) = \dfrac{2s}{s^2 + 2s + 5}e^{-2s}$

▶ **Solution.** Let $G_1(s) = \dfrac{2s}{s^2+2s+5}$. Then

$$\begin{aligned} G_1(s) = \frac{2s}{s^2+2s+5} &= \frac{2s}{(s+1)^2+4} \\ &= \frac{2(s+1)-2}{(s+1)^2+4} \\ &= \frac{2(s+1)}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}. \end{aligned}$$

Thus, taking the inverse Laplace transform gives

$$g_1(t) = \mathcal{L}^{-1}\{G_1(s)\} = 2e^{-t}\cos 2t - e^{-t}\sin 2t,$$

so that the inverse of the second translation theorem gives

$$\begin{aligned} g(t) = \mathcal{L}^{-1}\{G(s)\} &= g_1(t-2)h(t-2) \\ &= \Big(2e^{-(t-2)}\cos 2(t-2) - e^{-(t-2)}\sin 2(t-2)\Big)h(t-2) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 2 \\ 2e^{-(t-2)}\cos 2(t-2) - e^{-(t-2)}\sin 2(t-2) & \text{if } t \ge 2. \end{cases} \end{aligned}$$

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5. [**20 Points**] Solve the following initial value problem:

$$y'' + y = \delta(t-\pi), \qquad y(0) = 0,\ y'(0) = 1.$$

(Remember that $\delta(t-c)$ is the Dirac delta function centered at c.)

Give a careful sketch of the graph of the solution for the interval $0 \le t \le 2\pi$.

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function. Apply the Laplace transform to both sides of the equation to get

$$s^2Y(s) - 1 + Y(s) = e^{-\pi s}.$$

Thus,

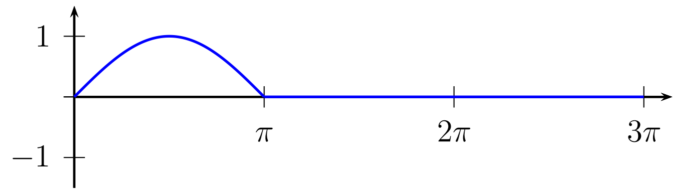
$$(s^2+1)Y(s) = 1 + e^{-\pi s},$$

and hence

$$Y(s) = \frac{1}{s^2+1} + e^{-\pi s}\frac{1}{s^2+1}.$$

Apply the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \sin t + h(t-\pi)\sin(t-\pi) \\ &= \sin t - (\sin t)h(t-\pi) \\ &= \begin{cases} \sin t & \text{if } 0 \le t < \pi, \\ 0 & \text{if } t \ge \pi. \end{cases} \end{aligned}$$



Graph of $y(t)$.

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Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$