**Instructions.** Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A table of Laplace transforms is appended to the exam.

1. [**10 Points**] Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$. Compute each of the following matrices, if it exists. If it does not exist, explain why.

   (a) $AB$ (b) $BA$ (c) $A^2$ (d) $B^2$

2. [**15 Points**] Find all solutions to the linear system

$$\begin{array}{rcrcrcr} x_1 & - & 6x_2 & - & 4x_3 & = & -5 \\ 2x_1 & - & 10x_2 & - & 9x_3 & = & -4 \\ -x_1 & + & 6x_2 & + & 5x_3 & = & 3 \end{array}$$

   Use Gauss-Jordan elimination.

3. [**15 Points**] Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \\ 0 & 4 & 2 \end{bmatrix}$.

   (a) Compute $\det A$.

   (b) Compute the inverse of $A$.

   (c) *Using your answer to part* (b), solve the linear system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}$.

   (a) [**3 Points**] Verify that $\det(sI - A) = s^2 + 4$.

   (b) [**10 Points**] Compute the matrix exponential $e^{At}$.

   (c) [**7 Points**] Find the general solution of $\mathbf{y}' = A\mathbf{y}$.

5. Consider the following first order linear system of differential equations

$$\begin{aligned} y_1' &= 7y_1 + y_2 \\ y_2' &= -4y_1 + 3y_2 \end{aligned}$$

   (a) [**3 Points**] Write the system in matrix form $\mathbf{y}' = A\mathbf{y}$.

   (b) [**3 Points**] Verify that $\det(sI - A) = (s - 5)^2$.

   (c) [**14 Points**] Solve this system with the initial conditions $y_1(0) = 1$, $y_2(0) = -2$.

## Laplace Transform Table

| | $f(t)$ | $\longleftrightarrow$ | $F(s) = \mathcal{L}\{f(t)\}(s)$ |
|---|---|---|---|
| 1. | $1$ | $\longleftrightarrow$ | $\dfrac{1}{s}$ |
| 2. | $t^n$ | $\longleftrightarrow$ | $\dfrac{n!}{s^{n+1}}$ |
| 3. | $e^{at}$ | $\longleftrightarrow$ | $\dfrac{1}{s-a}$ |
| 4. | $t^n e^{at}$ | $\longleftrightarrow$ | $\dfrac{n!}{(s-a)^{n+1}}$ |
| 5. | $\cos bt$ | $\longleftrightarrow$ | $\dfrac{s}{s^2+b^2}$ |
| 6. | $\sin bt$ | $\longleftrightarrow$ | $\dfrac{b}{s^2+b^2}$ |
| 7. | $e^{at}\cos bt$ | $\longleftrightarrow$ | $\dfrac{s-a}{(s-a)^2+b^2}$ |
| 8. | $e^{at}\sin bt$ | $\longleftrightarrow$ | $\dfrac{b}{(s-a)^2+b^2}$ |
| 9. | $h(t-c)$ | $\longleftrightarrow$ | $\dfrac{e^{-sc}}{s}$ |
| 10. | $\delta_c(t) = \delta(t-c)$ | $\longleftrightarrow$ | $e^{-sc}$ |

## Laplace Transform Principles

| | | | |
|---|---|---|---|
| **Linearity** | $\mathcal{L}\{af(t)+bg(t)\}$ | $=$ | $a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$ |
| **Input Derivative Principles** | $\mathcal{L}\{f'(t)\}(s)$ | $=$ | $s\mathcal{L}\{f(t)\}-f(0)$ |
| | $\mathcal{L}\{f''(t)\}(s)$ | $=$ | $s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$ |
| **First Translation Principle** | $\mathcal{L}\{e^{at}f(t)\}$ | $=$ | $F(s-a)$ |
| **Transform Derivative Principle** | $\mathcal{L}\{-tf(t)\}(s)$ | $=$ | $\dfrac{d}{ds}F(s)$ |
| **Second Translation Principle** | $\mathcal{L}\{h(t-c)f(t-c)\}$ | $=$ | $e^{-sc}F(s)$, or |
| | $\mathcal{L}\{g(t)h(t-c)\}$ | $=$ | $e^{-sc}\mathcal{L}\{g(t+c)\}$. |