Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A table of Laplace transforms is appended to the exam.

1. [**10 Points**] Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$. Compute each of the following matrices, if it exists. If it does not exist, explain why.

   (a) AB (b) BA (c) A^2 (d) B^2

   ▶ **Solution.** (a) $AB = \begin{bmatrix} -7 & 7 \\ 4 & 1 \\ -6 & -5 \end{bmatrix}$

   (b) BA does not exist since the number of columns of B is 2, which is different from the number of rows of A.

   (c) A^2 does not exist since A is not square.

   (d) $B^2 = \begin{bmatrix} -5 & 4 \\ -6 & -5 \end{bmatrix}$

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2. [**15 Points**] Find all solutions to the linear system

$$\begin{array}{rcrcrcr} x_1 & - & 6x_2 & - & 4x_3 & = & -5 \\ 2x_1 & - & 10x_2 & - & 9x_3 & = & -4 \\ -x_1 & + & 6x_2 & + & 5x_3 & = & 3 \end{array}$$

   Use Gauss-Jordan elimination.

   ▶ **Solution.** The augmented matrix corresponding to the given system is

$$\begin{bmatrix} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{bmatrix}.$$

   Apply Gauss-Jordan elimination to the augmented matrix:

$$\begin{bmatrix} 1 & -6 & -4 & -5 \\ 2 & -10 & -9 & -4 \\ -1 & 6 & 5 & 3 \end{bmatrix} \xrightarrow[R_1+R-3\to R_3]{-2R_1+R_2\to R_2} \begin{bmatrix} 1 & -6 & -4 & -5 \\ 0 & 2 & -1 & 6 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3+R_2\to R_2]{3R_2+R_1\to R_1} \begin{bmatrix} 1 & 0 & -7 & 13 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix} \xrightarrow[(1/2)R_2\to R_2]{7R_3+R_1\to R_1} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

This matrix is in reduced row echelon form and the solution of the original system can be read off as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}.$$

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3. **[15 Points]** Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & 1 \\ 0 & 4 & 2 \end{bmatrix}$.

(a) Compute $\det A$.

▶ **Solution.** Use cofactor expansion along the first row:

$$\begin{aligned} \det A &= 1 \cdot \det \begin{bmatrix} -1 & 1 \\ 4 & 2 \end{bmatrix} + (-1)^{1+2}(-2) \det \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} + (-1)^{1+3} 0 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 4 \end{bmatrix} \\ &= ((-1)2 - 1 \cdot 4) + 2 \cdot (2 \cdot 2 - 0 \cdot 1) + 0 \cdot (2 \cdot 4 - 0(-1)) \\ &= -6 + 8 = 2. \end{aligned}$$

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(b) Compute the inverse of A.

▶ **Solution.** Use row reduction of the augmented matrix $[A \;\; I]$:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 1 \\ 2 & -1 & 1 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-2)R_1+R_2 \to R_2} \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -2 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{-R_2+R_3 \to R_3} \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow[2R_3+R_1 \to R_1]{-3R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 2 & 5 & -2 & 2 \\ 0 & 0 & -2 & -8 & 4 & -3 \\ 0 & 1 & 1 & 2 & -1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 2 & 5 & -2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & -2 & -8 & 4 & -3 \end{bmatrix} \xrightarrow{(1/2)R_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 5 & -2 & 2 \\ 0 & 1 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 4 & -2 & \frac{3}{2} \end{bmatrix} \xrightarrow[22R_3+R_1 \to R_1]{-R_3+R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 & -3 & 2 & -1 \\ 0 & 1 & 0 & -2 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 & 4 & -2 & \frac{3}{2} \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} -3 & 2 & -1 \\ -2 & 1 & -\frac{1}{2} \\ 4 & -2 & \frac{3}{2} \end{bmatrix}.$$

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(c) *Using your answer to part* (b), solve the linear system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

▶ **Solution.** The solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 2 & -1 \\ -2 & 1 & -\frac{1}{2} \\ 4 & -2 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -\frac{9}{2} \\ \frac{19}{2} \end{bmatrix}.$$

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4. Let $A = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}$.

(a) [**3 Points**] Verify that $\det(sI - A) = s^2 + 4$.

▶ **Solution.**

$$\det(sI - A) = \det \begin{bmatrix} s+2 & -1 \\ 8 & s-2 \end{bmatrix} = (s+2)(s-2) - (-1)8$$
$$= s^2 - 4 + 8 = s^2 + 4.$$

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(b) [**10 Points**] Compute the matrix exponential e^{At}.

▶ **Solution.** First Since $\det(sI - A) = s^2 + 4$,

$$(sI - A)^{-1} = \frac{1}{s^2+4} \begin{bmatrix} s-2 & 1 \\ -8 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{s-2}{s^2+4} & \frac{1}{s^2+4} \\ \frac{-8}{s^2+4} & \frac{s+2}{s^2+4} \end{bmatrix}.$$

Thus,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} \cos 2t - \sin 2t & \frac{1}{2}\sin 2t \\ -4\sin 2t & \cos 2t + \sin 2t \end{bmatrix}.$$

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(c) [**7 Points**] Find the general solution of $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.** The general solution is $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$, where $\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where c_1 and c_2 are arbitrary real numbers. Thus, the general solution is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \cos 2t - \sin 2t & \frac{1}{2}\sin 2t \\ -4\sin 2t & \cos 2t + \sin 2t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$
$$= \begin{bmatrix} c_1(\cos 2t - \sin 2t) + c_2\frac{1}{2}\sin 2t \\ -4c_1\sin 2t + c_2(\cos 2t + \sin 2t) \end{bmatrix}$$

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5. Consider the following first order linear system of differential equations

$$\begin{aligned} y_1' &= 7y_1 + y_2 \\ y_2' &= -4y_1 + 3y_2 \end{aligned}$$

(a) [**3 Points**] Write the system in matrix form $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.**

$$\mathbf{y}' = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \mathbf{y} = A\mathbf{y},$$

where $A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$ ◀

(b) [**3 Points**] Verify that $\det(sI - A) = (s-5)^2$.

▶ **Solution.**

$$\begin{aligned} \det(sI - A) = \det \begin{bmatrix} s-7 & -1 \\ 4 & s-3 \end{bmatrix} &= (s-7)(s-3) - (-1)4 \\ &= s^2 - 10s + 21 + 4 = s^2 - 10s + 25 = (s-5)^2. \end{aligned}$$

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(c) [**14 Points**] Solve this system with the initial conditions $y_1(0) = 1$, $y_2(0) = -2$.

▶ **Solution.** First compute e^{At}: $sI - A = \begin{bmatrix} s-7 & -1 \\ 4 & s-3 \end{bmatrix}$, so

$$(sI-A)^{-1} = \frac{1}{(s-5)^2} \begin{bmatrix} s-3 & 1 \\ -4 & s-7 \end{bmatrix} = \begin{bmatrix} \frac{s-3}{(s-5)^2} & \frac{1}{(s-5)^2} \\ \frac{-4}{(s-5)^2} & \frac{s-7}{(s-5)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s-5} + \frac{2}{(s-5)^2} & \frac{1}{(s-5)^2} \\ \frac{-4}{(s-5)^2} & \frac{1}{s-5} - \frac{2}{(s-5)^2} \end{bmatrix}.$$

Thus,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} e^{5t} + 2te^{5t} & te^{5t} \\ -4te^{5t} & e^{5t} - 2te^{5t} \end{bmatrix},$$

and

$$\begin{aligned} \mathbf{y}(t) &= e^{At} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} + 2te^{5t} & te^{5t} \\ -4te^{5t} & e^{5t} - 2te^{5t} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} \\ -2e^{5t} \end{bmatrix}. \end{aligned}$$

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Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.