Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the main partial fraction decomposition theorem have been appended to the exam.

In Exercises 1 – 8, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' + 3y = 3e^{2t} - 2e^{-3t}, \quad y(0) = 2.$

   ► **Solution.** This equation is linear with $p(t) = 3$ so that an integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{3t}$. Multiplying the equation by e^{3t} gives

   $$e^{3t}y' + 3e^{3t}y = 3e^{5t} - 2.$$

   Thus,

   $$\frac{d}{dt}\left(e^{3t}y\right) = 3e^{5t} - 2,$$

   and integrating gives

   $$e^{3t}y = (3/5)e^{5t} - 2t + C.$$

   Hence, solving for y gives $y = (3/5)e^{2t} - 2te^{-3t} + Ce^{-3t}$. Now apply the initial condition $y(0) = 2$ to get $2 = y(0) = 3/5 + C$, which gives $C = \frac{7}{5}$. Therefore,

   $$\boxed{y(t) = \frac{3}{5}e^{2t} - 2te^{-3t} + \frac{7}{5}e^{-3t}.}$$

   ◄

2. [**12 Points**] $y' = -t/y, \quad y(0) = 4.$

   ► **Solution.** This equation is separable. Separate the variables to get $yy' = -t$. Write in differential form and integrate to get:

   $$\int y\,dy = -\int t\,dt.$$

   Integrating gives $y^2/2 = -t^2/2 + C$ and using the initial condition $y = 4$ when $t = 0$ gives $C = 8$. Hence, solving for y give

   $$\boxed{y(t) = \sqrt{-t^2 + 16}.}$$

   ◄

3. [**10 Points**] $y'' + 10y' + 29y = 0.$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 10s + 29 = (s+5)^2 + 4$, which has roots $-5 \pm 2i$. Thus, the solutions are given by

$$\boxed{y(t) = c_1 e^{-5t} \cos 2t + c_2 e^{-5t} \sin 2t.}$$

◀

4. [**10 Points**] $4y'' + 12y' + 9y = 0$.

▶ **Solution.** The characteristic polynomial is $q(s) = 4s^2 + 12s + 9 = (2s+3)^2$, which has a single root $-3/2$ of multiplicity 2. Thus, the solutions are given by

$$\boxed{y(t) = c_1 e^{-3t/2} + c_2 t e^{-3s/2}.}$$

◀

5. [**12 Points**] $y'' - 4y' + 3y = 0 = 0, \quad y(0) = 1,\ y'(0) = -1$.

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 - 4s + 3 = (s-3)(s-1)$, which has roots 1 and 3. Thus, the general solution of the differential equation is $y(t) = c_1 e^t + c_2 e^{3t}$. The initial conditions give the equations for c_1, c_2:

$$\begin{aligned} 1 &= y(0) = c_1 + c_2 \\ -1 &= y'(0) = c_1 + 3c_2. \end{aligned}$$

Solve these equations to get $c_1 = 2$, $c_2 = -1$. Thus, the solution of the initial value problem is

$$\boxed{y(t) = 2e^t - e^{3t}.}$$

◀

6. [**12 Points**] $y'' + 2y' - 8y = 4e^{2t}$.

▶ **Solution.** Use the method of undetermined coefficients. The characteristic polynomial of the associated homogeneous equation is $q(s) = s^2 + 2s - 8 = (s+4)(s-2)$. The roots are -4 and 2 so that the standard basis is $\mathcal{B}_q = \{e^{-4t}, e^{2t}\}$. The Laplace transform of the right hand side is $\mathcal{L}\{4e^{2t}\} = 4/(s-2)$, which has denominator $v(s) = s-2$. Thus, $q(s)v(s) = (s+4)(s-2)^2$ and

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-4t}, e^{2t}, te^{2t}\right\} \setminus \left\{e^{-4t}, e^{2t}\right\} = \left\{te^{2t}\right\}.$$

Thus, a particular solution of the nonhomogeneous equation will have the form $y_p(t) = Ate^{2t}$, and the unknown constant A can be determined by substituting this back in the differential equation. Compute the derivatives of $y_p(t)$:

$$y_p(t) = Ate^{2t}, \qquad y_p'(t) = A(e^{2t} + 2te^{2t}), \qquad y_p''(t) = A(4e^{2t} + 4te^{2t}).$$

Substituting in the differential equation gives

$$\begin{aligned} 4e^{2t} &= y_p'' + 2y_p' - 8y_p \\ &= A(4e^{2t} + 4te^{2t}) + 2A(e^{2t} + 2te^{2t}) - 8Ate^{2t} \\ &= 6Ae^{2t}. \end{aligned}$$

Solving for A gives $A = 2/3$. Thus, $y_p(t) = (2/3)te^{2t}$ and the general solution is

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1e^{2t} + c_2e^{-4t} + \frac{2}{3}te^{2t}.}$$

◀

7. [**10 Points**] $2t^2y'' + 5ty' - 2y = 0$.

▶ **Solution.** This is a Cauchy-Euler equation with indicial polynomial

$$q(s) = 2s(s-1) + 5s - 2 = 2s^3 + 3s - 2 = (2s-1)(s+2),$$

which has roots $1/2$ and -2. Thus, the general solution is

$$\boxed{y(t) = c_1t^{1/2} + c_2t^{-2}.}$$

◀

8. [**12 Points**] $y'' + 25y = 2\delta(t-\pi)$, $\quad y(0) = 2$, $y'(0) = 3$. (Recall that $\delta(t)$ is the Dirac delta function.)

▶ **Solution.** Use the Laplace transform method. Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the unknown solution of the initial value problem. Applying the Laplace transform to the differential equation gives:

$$s^2Y(s) - 2s - 3 + 25Y(s) = 2e^{-\pi s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{2s+3}{s^2+25} + \frac{2}{s^2+25}e^{-\pi s}.$$

Then take the inverse laplace transform to get:

$$\boxed{y(t) = 2\cos 5t + \frac{3}{5}\sin 5t + \frac{2}{5}h(t-\pi)\sin 5(t-\pi).}$$

◀

9. [**12 Points**] Find a particular solution of the differential equation

$$y'' + \frac{2}{t}y' - \frac{6}{t^2}y = 80t^3,$$

given the fact that the general solution of the associated homogeneous equation is

$$y_h = c_1t^2 + c_2t^{-3}.$$

▶ **Solution.** Use variation of parameters. A particular solution is given by

$$y_p = u_1t^2 + u_2t^{-3}$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1't^2 + u_2't^{-3} &= 0 \\ 2u_1't - 3u_2't^{-4} &= 80t^3. \end{aligned}$$

Multiplying the first equation by $3t^{-1}$ and adding to the second eliminates u_2', giving $5u_1't = 80t^3$, or $u_1' = 16t^2$. Substituting in the first equation then gives

$$u_2' = -u_1't^5 = -16t^7.$$

Integrating then gives

$$u_1 = \frac{16}{3}t^3 \quad \text{and} \quad u_2 = -2t^8,$$

which gives

$$y_p = \frac{16}{3}t^3t^2 - 2t^8t^{-3} = \frac{10}{3}t^5.$$

◀

10. [**8 Points**] Find the Laplace transform $F(s)$ of the following function $f(t)$.

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 3, \\ t^2 - 9 & \text{if } t \ge 3. \end{cases}$$

▶ **Solution.** First write $f(t)$ in terms of the heaviside functions:

$$\begin{aligned} f(t) &= 2\chi_{[0,3)}(t) + (t^2 - 9)\chi_{[3,\infty)}(t) \\ &= 2 - 2h(t-3) + (t^2 - 9)h(t-3) \\ &= 2 + (t^2 - 11)h(t-3). \end{aligned}$$

Then

$$F(s) = \frac{2}{s} + e^{-3s}\mathcal{L}\left\{(t+3)^2 - 11\right\}$$
$$= \frac{2}{s} + e^{-3s}\mathcal{L}\left\{t^2 + 6s - 2\right\}.$$

Thus,

$$\boxed{F(s) = \frac{2}{s} + e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} - \frac{2}{s}\right).}$$

◀

11. [**14 Points**] Compute each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{1-2s}{(s-3)^2+5}\right\}$

▶ **Solution.** Since,

$$\begin{aligned}\frac{1-2s}{(s-3)^2+5} &= \frac{1}{(s-3)^2+5} - \frac{2s}{(s-3)^2+5}\\ &= \frac{1}{(s-3)^2+5} - \frac{2(s-3)+6}{(s-3)^2+5}\\ &= \frac{-5}{(s-3)^2+5} - \frac{2(s-3)}{(s-3)^2+5},\end{aligned}$$

we have

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1-2s}{(s-3)^2+5}\right\} = \frac{-5}{\sqrt{5}}e^{3t}\sin\sqrt{5}t - 2e^{3t}\cos\sqrt{5}t.}$$

◀

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{(s+1)(s^2-4)}\right\}$

▶ **Solution.** Use partial fractions to write

$$\frac{4s}{(s+1)(s+2)(s-2)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s-2},$$

where

$$A = \left.\frac{4s}{(s+2)(s-2)}\right|_{s=-1} = \frac{-4}{-3} = \frac{4}{3}$$
$$B = \left.\frac{4s}{(s+1)(s-2)}\right|_{s=-2} = \frac{-8}{(-1)(-4)} = -2$$
$$C = \left.\frac{4s}{(s+1)(s+2)}\right|_{s=2} = \frac{8}{(3)(4)} = \frac{2}{3}$$

so that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{4s}{(s+1)(s^2-4)}\right\} = \frac{4}{3}e^{-t} - 2e^{-2t} + \frac{2}{3}e^{2t}.}$$

◀

12. [**18 Points**] Let $A = \begin{bmatrix} 2 & 4 \\ -4 & -6 \end{bmatrix}$.

(a) Compute $(sI - A)^{-1}$.

▶ **Solution.** $(sI - A) = \begin{bmatrix} s-2 & -4 \\ 4 & s+6 \end{bmatrix}$ so

$$p(s) = \det(sI - A) = (s-2)(s+6) + 16 = s^2 + 4s + 4 = (s+2)^2.$$

Hence

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+6}{(s+2)^2} & \frac{4}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{s-2}{(s+2)^2} \end{bmatrix}.$$

◀

(b) Find $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

▶ **Solution.**

$$\begin{aligned}
\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} &= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{s+6}{(s+2)^2} & \frac{4}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{s-2}{(s+2)^2} \end{bmatrix}\right\} \\
&= \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{(s+2)+4}{(s+2)^2} & \frac{2}{(s+2)^2} \\ \frac{-4}{(s+2)^2} & \frac{(s+2)-4}{(s+2)^2} \end{bmatrix}\right\} \\
&= \begin{bmatrix} e^{-2t} + 4te^{2t} & 4te^{-2t} \\ -4te^{-2t} & e^{-2t} - 4te^{-2t} \end{bmatrix}
\end{aligned}$$

◀

(c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.** If $\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$
\begin{aligned}
\mathbf{y}(t) &= \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\mathbf{y}(0) = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} e^{-2t} + 4te^{2t} & 4te^{-2t} \\ -4te^{-2t} & e^{-2t} - 4te^{-2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1e^{-2t} + (4c_2 + 4c_1)te^{-2t} \\ c_2e^{-2t} - (4c_2 + 4c_1)te^{-2t} \end{bmatrix}.
\end{aligned}
$$

◀

(d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

▶ **Solution.** Let $c_1 = -2$ and $c_2 = 3$ in Part (c):

$$\mathbf{y}(t) = \begin{bmatrix} -2e^{-2t} + 4te^{-2t} \\ 3e^{-2t} - 4te^{-2t} \end{bmatrix}.$$

◀

13. [**12 Points**] A tank initially contains 2000 gallons of water with 100 pounds of salt dissolved. Brine (a water-salt mixture) containing 0.8 pounds of salt per gallon flows into the tank at the rate of 5 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the same rate of 5 gal/min.

(a) Find the amount of salt $y(t)$ in the tank at time t.

▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0.8 lb/gal × 5 gal/min, i.e., 4 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 5.$$

Since mixture is entering and leaving at the same volume rate of 5 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 2000$. Hence $y(t)$ satisfies the equation

$$y' = 4 - \frac{5}{2000}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$y' + \frac{1}{400}y = 4, \qquad y(0) = 100.$$

This is a linear differential equation with integrating factor $\mu(t) = e^{t/400}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{t/400}y\right) = 4e^{t/400}.$$

Integration of this equation gives

$$e^{t/400}y = 1600e^{t/400} + C,$$

where C is an integration constant. Dividing by $e^{t/400}$ gives

$$y = 1600 + Ce^{-t/400},$$

and the initial condition $y(0) = 100$ gives a value of $C = --1500$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 1600 - 1500e^{-t/400}.}$$

◀

(b) How much salt does the tank contain after 3 hours?

▶ **Solution.** At 3 hours the time is $t = 3 \times 60 = 180$ min. Thus,

$$\begin{aligned} y(180) &= 1600 - 1500e^{-180/400} \\ &= 1600 - 1500e^{-9/20} \\ &\approx 643.56 \text{ lbs.} \end{aligned}$$

◀

(c) What is $\lim_{t\to\infty} y(t)$?

▶ **Solution.** $\lim_{t\to\infty} y(t) = 1600$. ◀

Laplace Transform Table

	$f(t)$		$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$