Exam 4 will be on Monday, November 26. The syllabus is Sections 8.1 -8.4, 9.1 - 9.3 from the text. You should review the assigned exercises in these sections. Following is a brief list (not necessarily complete) of terms, skills, and formulas with which you should be familiar.

- You should know how to do the algebraic operations on matrices. Addition, scalar multiplication, and multiplication of matrices is in Section 8.1.
- The following two facts for 2×2 matrices are fundamental. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\det A = ad - bd \neq 0 \iff A \text{ is invertible}$$

and

$$\text{if } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{ad - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- You should know how to row reduce a matrix to reduced row echelon form.
- Know how to use Gauss-Jordan elimination to solve a system of linear equations.
- Know how to find the inverse of a matrix via Gauss-Jordan elimination, and also by means of the Adjoint formula for the inverse (Corollary 11, Page 611).
- Know how to calculate determinants of square matrices by elementary row and column operation and also by the Laplace expansion along a row or a column.
- Know the relationship between determinants and invertible matrices.
- For a constant coefficient homogeneous linear system

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

the unique solution is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0,$$

where the matrix exponential e^{At} is *defined* by the infinite series

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots.$$

Know how to compute e^{At} from the definition for some simple 2×2 matrices, as is done in the examples on Pages 650–1.

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is a 2×2 matrix then the system of differential equations

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

which can also be written as

$$\begin{aligned} y_1' &= ay_1 + by_2 \\ y_2' &= cy_1 + dy_2 \end{aligned} \qquad y_1(0) = c_1, \ y_2(0) = c_2,$$

has the solution

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0).$$

Since

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\},$$

(see Theorem 4, Page 652) the following algorithm can be used to solve $\boldsymbol{y}' = A\boldsymbol{y}$ when the matrix A is a constant 2×2 matrix.

Algorithm 1.

1. Form the matrix $sI - A = \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}$.
2. Compute the characteristic polynomial
$$p(s) = \det(sI - A) = \det \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix} = s^2 - (a+d)s + (ad - bd).$$
3. Compute
$$(sI - A)^{-1} = \frac{1}{p(s)} \begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} = \begin{bmatrix} \dfrac{s-d}{p(s)} & \dfrac{b}{p(s)} \\ \dfrac{c}{p(s)} & \dfrac{s-a}{p(s)} \end{bmatrix}.$$
4. Compute
$$\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{s-d}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{b}{p(s)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{c}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{s-a}{p(s)}\right\} \end{bmatrix} = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix}.$$
5. The solution $\boldsymbol{y}(t)$ is then
$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) = \left(\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\right)\boldsymbol{y}(0) = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 h_1(t) + c_2 h_2(t) \\ c_1 h_3(t) + c_2 h_4(t) \end{bmatrix}.$$

The following is a small set of exercises of types identical to those already assigned.

1. Let $A = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Determine all values of x for which $AB = BA$.
2. Let A be a 3×1 matrix, B a 2×3 matrix, and C a 2×1 matrix. which of the following matrix products is well defined and the result is a 1×2 matrix?
   (a) AB
   (b) $B^T C$
   (c) $C^T BA$
   (d) BAC^T
   (e) $(BA)^T$
3. Find all solutions of the following system of linear equations. Be sure to show all your steps!
$$\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & = & 8 \\ x_1 & + & 3x_2 & & & + & x_4 & = & 7 \\ x_1 & & & + & 2x_3 & + & x_4 & = & 3 \end{array}$$

4. Let $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$.

   (a) Compute A^{-1}.

   (b) Using your answer to part (a), solve the system of equations

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & & & = & 1 \\ 2x_1 & & & + & x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 1 \end{array}$$

5. Find conditions that b_1, b_2, and b_3 must satisfy for the following system to be consistent:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & - & 3x_3 & = & b_1 \\ 2x_1 & + & 3x_2 & + & 3x_3 & = & b_2 \\ 5x_1 & + & 9x_2 & - & 6x_3 & = & b_3 \end{array}$$

6. (a) Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & k & -1 \\ 1 & k^2 & 1 \end{bmatrix}.$$

   (b) Find all the values of k for which A is **not** invertible.

7. Suppose that A is the 3×3 matrix

$$\begin{bmatrix} 1 & 8 & 3 \\ x & y & z \\ -3 & 7 & 2 \end{bmatrix}.$$

   Assuming that $\det(A) = 5$, compute the determinant of the following matrices:

$$\text{(a) } B = 2A^2; \qquad \text{(b) } C = \begin{bmatrix} x & y & z \\ 1 & 8 & 3 \\ 6+4x & -14+4y & -4+4z \end{bmatrix}.$$

8. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

   (a) Compute $(sI - A)$ and $(sI - A)^{-1}$.

   (b) Find $\mathcal{L}^{-1}((sI - A)^{-1})$.

   (c) What is e^{At}?

   (d) Solve the system $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

9. Solve the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$ where $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

10. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

11. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -3 \\ 3 & 6 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

12. Let $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$.

    (a) Compute A^2, A^3, etc.

    (b) Compute e^{At} from the definition as a series.

    (c) Solve the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Answers

1. ▶ **Solution.** $AB = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8+x & 16 \\ 8 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 14 & x+10 \\ 8 & x \end{bmatrix}$. These matrices are equal if $8+x=14$, $16=x+10$, $8=8$, and $6=x$. These are all satisfied for $\boxed{x=6}$. ◀

2. $\boxed{(BA)^T}$

3. ▶ **Solution.** Use Gauss-Jordan elimination on the augmented matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & -2 & -1 & 0 & -5 \end{bmatrix}$$

$$\begin{matrix} \\ \longrightarrow \\ R_3 \mapsto R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & -7 & 0 & -7 \end{bmatrix} \begin{matrix} -\frac{1}{7}R_2 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \mapsto R_2 + 3R_3 \\ \longrightarrow \\ R_1 \mapsto R_1 - 3R_3 \end{matrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 \mapsto R_1 - 2R_2 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row-echelon form and is the augmented matrix of the linear system

$$\begin{array}{ccccccc} x_1 & & & + & x_4 & = & 1 \\ & x_2 & & & & = & 2 \\ & & x_3 & & & = & 1 \end{array}$$

The solution set of this last system is the same as that of the original system and is given by:

$$\mathcal{S} = \{(1 - x_4, 2, 1, x_4) : \; x_4 \text{ is arbitrary}\}.$$

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4. (a) ▶ **Solution.** Use Gauss-Jordan elimination of the augmented matrix $\begin{bmatrix} A & | & I_3 \end{bmatrix}$:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_2 \leftrightarrow R_1 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{bmatrix} \begin{matrix} R_3 \mapsto R_3 - 2R_2 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 & 1 & -2 \end{bmatrix}$$

$$\begin{matrix} \frac{1}{3}R_3 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 + R_3 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\begin{matrix} R_1 \mapsto R_1 + R_2 \\ \longrightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

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(b) ▶ **Solution.**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

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5. ▶ **Solution.** Start by trying to row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & b_1 \\ 2 & 3 & 3 & b_2 \\ 5 & 9 & -6 & b_3 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - 5R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & -1 & 9 & b_2 - 2b_1 \\ 0 & -1 & 9 & b_3 - 5b_1 \end{bmatrix}$$

$$\begin{matrix} -R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & 1 & 9 & -b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 3b_1 - b_2 \end{bmatrix}$$

In order for the system to be consistent, the last entry in the last column must be 0, and this entry being 0 is also sufficient to be able to complete the row reduction and solve the system. Hence, the system is consistent if and only if $\boxed{3b_1 + b_2 = b_3}$. ◀

6. ▶ **Solution.** (a) Use cofactor expansion along the second column to get

$$|A| = (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - k^2\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 + 2k^2.$$

(b) A is not invertible if and only if $\det A = 0$ which is true if and only if $k^2 - 1 = 0 \iff k = \pm 1$.

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7. ▶ **Solution.** (a) $\det B = 2^3 \det A^2 = 2^3(\det A)^2 = 8 \times 5^2 = 200$.
(b) C is obtained from A by the following sequence of elementary row operations:

$$-2R_3; \qquad R_3 \mapsto R_3 + 4R_1; \qquad R_1 \leftrightarrow R_2.$$

The first operation multiplies the determinant by -2, the second does not change the determinant, and the third multiplies the determinant by -1. Thus, the determinant of the final matrix C is given by:

$$\det C = (-2)(-1)\det A = 10.$$

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8. $sI - A = \begin{bmatrix} s-1 & 2 \\ 3 & s-2 \end{bmatrix}$; $(sI - A)^{-1} = \begin{bmatrix} \dfrac{s-2}{(s-4)(s+1)} & \dfrac{-2}{(s-4)(s+1)} \\ \dfrac{-3}{(s-4)(s+1)} & \dfrac{s-1}{(s-4)(s+1)} \end{bmatrix}$

(b) $\frac{1}{5}\begin{bmatrix} 2e^{4t} + 3e^{-t} & -2e^{4t} + 2e^{-t} \\ -3e^{4t} + 3e^{-t} & -3e^{4t} + 2e^{-t} \end{bmatrix}$ (c) e^{At} is same as $\mathcal{L}^{-1}((sI - A)^{-1})$.

(d) $\boldsymbol{y}(t) = \frac{1}{5}\begin{bmatrix} -8e^{4t} + 3e^{-t} \\ -6e^{4t} + 3e^{-t} \end{bmatrix}$

9. $\boldsymbol{y}(t) = \frac{1}{6}\begin{bmatrix} (5c_1 - c_2)e^{4t} + (c_1 + c_1)e^{-2t} \\ (-5c_1 + c_2)e^{4t} + (5c_2 + 5c_1)e^{-2t} \end{bmatrix}$

10. $\boldsymbol{y}(t) = \frac{1}{2}\begin{bmatrix} 1 + e^{4t} \\ -2 + 2e^{4t} \end{bmatrix}$

11. $\boldsymbol{y}(t) = \begin{bmatrix} e^{3t} + 3te^{3t} \\ -2e^{3t} - 3te^{3t} \end{bmatrix}$

12. (a) $A^2 = A^3 = \cdots = A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $n \geq 2$.

(b) $e^{At} = I + At = \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}$

(c) $\boldsymbol{y}(t) = \begin{bmatrix} 3-2t \\ -2+t \end{bmatrix}$.