Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A copy of the table of Laplace transforms and convolution products from the text will be supplied, and these tables can be used for all problems.

1. [24 Points] Compute the Laplace transform of each of the following functions.

   (a) $f_1(t) = t^4 e^{-t/2}$

   ▶ **Solution.** Use formula C.2.6 with $n - 1 = 4$ and $a = -1/2$ to get

   $$\boxed{F_1(s) = \frac{4!}{\left(s + \frac{1}{2}\right)^5}.}$$

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   (b) $f_2(t) = e^{2t}(3\cos 5t + 5\sin 3t)$

   ▶ **Solution.** $f_2(t) = 3e^{2t}\cos 5t + 5e^{2t}\sin 3t$ so apply C.2.9 and C.2.10 to get

   $$\boxed{F_2(s) = \frac{3(s-2)}{(s-2)^2 + 25} + \frac{15}{(s-2)^2 + 9}.}$$

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   (c) $f_3(t) = (2t + 1)(t^2 + 3)$

   ▶ **Solution.** Expand $f_3(t)$ to get $f_3(t) = 2t^3 + t^2 + 6t + 3$ and apply C.2.3 to get

   $$\boxed{F_3(s) = \frac{12}{s^4} + \frac{2}{s^3} + \frac{6}{s^2} + \frac{3}{s} = \frac{12 + 2s + 6s^2 + 3s^3}{s^4}.}$$

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   (d) $f_4(t) = e^{5t} * \sin \pi t = \int_0^t e^{5x} \sin \pi(t - x)\, dx$

   ▶ **Solution.** Apply formula C.1.13 (Laplace transform of the convolution product) to get

   $$\boxed{F_4(s) = \frac{1}{(s-5)}\frac{\pi}{(s^2 + \pi^2)} = \frac{\pi}{(s-5)(s^2 + \pi^2)}.}$$

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2. [10 Points] Find the Laplace transform of the solution $y(t)$ of the initial value problem

   $$2y'' - y' + 4y = 7\cos 3t, \qquad y(0) = -2,\ y'(0) = 4.$$

   Note that you are only asked to find the Laplace transform $Y(s)$ of $y(t)$, not $y(t)$ itself.

▶ **Solution.** Apply the Laplace transform to both sides of the differential equation using C.1.5, C.1.6, and C.2.8, to get

$$2(s^2Y + 2s - 4) - (sY + 2) + 4Y = \frac{7s}{s^2+9},$$

so that

$$(2s^2 - s + 4)Y + 4s - 8 - 2 = \frac{7s}{s^2+9}.$$

Solve for Y to get

$$\boxed{Y = \frac{-4s+10}{2s^2 - s + 4} + \frac{7s}{(2s^2 - s + 4)(s^2+9)}.}$$

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3. [24 Points] Compute the inverse Laplace transform of each of the following functions.

(a) $F(s) = \dfrac{2s+3}{s^2+s-12}$

▶ **Solution.** Since $s^2 + s - 12 = (s+4)(s-3)$, we can expand $F(s)$ into partial fractions as

$$F(s) = \frac{A}{s+4} + \frac{B}{s-3}$$

where

$$A = \left.\frac{2s+3}{s-3}\right|_{s=-4} = \frac{5}{7} \quad \text{and} \quad B = \left.\frac{2s+3}{s+4}\right|_{s=3} = \frac{9}{7}.$$

Hence,

$$F(s) = \frac{5/7}{s+4} + \frac{9/7}{s-3},$$

so that formula C.2.4 gives

$$\boxed{f(t) = \frac{5}{7}e^{-4t} + \frac{9}{7}e^{3t}.}$$

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(b) $G(s) = \dfrac{s^3+s^2+4}{s^2(s^2+2)}$

▶ **Solution.** Expand $G(s)$ into a partial fraction using the algorithm presented in class as follows.

$$G(s) = \frac{s^3+s^2+4}{s^2(s^2+2)} = \frac{A}{s^2} + \frac{T(s)}{s(s^2+2)}$$

where

$$A = \left.\frac{s^3+s^2+4}{s^2+2}\right|_{s=0} = 2$$

and $T(s)$ is computed from the equation

$$sT(s) = (s^3 + s^2 + 4) - 2(s^2 + 2) = s^3 - s^2 = s(s^2 - s),$$

so that $T(s) = s^2 - s$. Then

$$\begin{aligned} G(s) &= \frac{2}{s^2} + \frac{s^2 - s}{s(s^2 + 2)} \\ &= \frac{2}{s^2} + \frac{s - 1}{s^2 + 2} \\ &= \frac{2}{s^2} + \frac{s}{s^2 + 2} - \frac{1}{s^2 + 2}. \end{aligned}$$

Now take the inverse Laplace transform using C.2.2, C.2.7, and C.2.8:

$$\boxed{g(t) = 2t + \cos\sqrt{2}t - \frac{1}{\sqrt{2}}\sin\sqrt{2}t.}$$

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(c) $H(s) = \dfrac{5s + 12}{s^2 + 6s + 13}$

▶ **Solution.** Complete the square to get $s^2 + 6s + 13 = (s + 3)^2 + 4$ and then rewrite the top of the fraction to get

$$\begin{aligned} H(s) &= \frac{5s + 12}{s^2 + 6s + 13} \\ &= \frac{5s + 12}{(s + 3)^2 + 4} \\ &= \frac{5(s + 3) - 3}{(s + 3)^2 + 4} \\ &= \frac{5(s + 3)}{(s + 3)^2 + 4} - \frac{3}{(s + 3)^2 + 4}. \end{aligned}$$

Now apply C.2.9 and C.2.10 to get

$$\boxed{h(t) = 5e^{-3t}\cos 2t - \frac{3}{2}e^{-3t}\sin 2t.}$$

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4. [21 Points] Find the characteristic polynomial and the general solution of each of the following constant coefficient linear homogeneous differential equations:

   (a) $4y'' - 4y' + y = 0$

▶ **Solution.** The characteristic polynomial is $p(s) = 4s^2 - 4s + 1 = (2s-1)^2$, which has a single root $1/2$ of multiplicity 2. Hence

$$\boxed{y = c_1 e^{t/2} + c_2 t e^{t/2}.}$$

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(b) $4y'' + 4y' - 3y = 0$

▶ **Solution.** The characteristic polynomial is $p(s) = 4s^2 + 4s - 3 = (2s+3)(2s-1)$ which the two distinct real roots $-3/2$ and $1/2$. Hence

$$\boxed{y = c_1 e^{-3t/2} + c_2 e^{t/2}.}$$

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(c) $y'' + 4y' + 9y = 0$

▶ **Solution.** The characteristic polynomial is $p(s) = s^2 + 4s + 9 = (s+2)^2 + 5$ which has the pair of complex conjugate roots $-2 \pm \sqrt{5}i$. Hence

$$\boxed{y = c_1 e^{-2t} \cos \sqrt{5}t + c_2 e^{-2t} \sin \sqrt{5}t.}$$

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5. [21 Points] Find the general solution of the constant coefficient homogeneous linear differential equation with the given characteristic polynomial $p(s)$.

(a) $p(s) = (s^2 - 5)^2 (s - 5)^3$

▶ **Solution.** $p(s) = (s - \sqrt{5})^2 (s + \sqrt{5})^2 (s - 5)^3$, so the roots of $p(s)$ are $\pm\sqrt{5}$, each of multiplicity 2 and 5 of multiplicity 3. Thus

$$\boxed{y = (c_1 + c_2 t) e^{\sqrt{5}t} + (c_3 + c_4 t) e^{-\sqrt{5}t} + (c_5 + c_6 t + c_7 t^2) e^{5t}.}$$

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(b) $p(s) = s^4 + 3s^2 - 4$

▶ **Solution.** $p(s) = (s^2 + 4)(s^2 - 1) = (s^2 + 4)(s + 1)(s - 1)$ so the roots of $p(s)$ are ± 1 and $\pm 2i$, each of multiplicity 1. Thus

$$\boxed{y = c_1 e^{t} + c_2 e^{-t} + c_3 \cos 2t + c_4 \sin 2t.}$$

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(c) $p(s)$ has degree 4 and has roots -2 with multiplicity 2 and $2 \pm 3i$, each with multiplicity 1.

▶ **Solution.**

$$\boxed{y = (c_1 + c_2 t) e^{-2t} + c_3 e^{2t} \cos 3t + c_4 e^{2t} \sin 3t.}$$

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