Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A copy of the Table of Laplace transforms from the text will be provided.

1. [**16 Points**] Solve: $2t^2y'' + 7ty' - 3y = 0$.

   ▶ **Solution.** This is a Cauchy-Euler equation with indicial polynomial

   $$q(s) = 2s(s-1) + 7s - 3 = 2s^2 + 5s - 3 = (2s-1)(s+3),$$

   which has the two distinct real roots $1/2$ and -3. Hence the general solution is

   $$\boxed{y = c_1 \left|t\right|^{1/2} + c_2 \left|t\right|^{-3}.}$$

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2. [**16 Points**] Solve: $y'' + y' - 6y = 3e^{2t}$.

   ▶ **Solution.** This equation could be solved either by the method of undetermined coefficients or variation of parameters, or it could be solved directly by use of Laplace transforms. I will use the method of undetermined coefficients.

   The characteristic equation is $p(s) = s^2 + s - 6 = (s-2)(s+3)$ and $F(s) = \mathcal{L}\left\{3e^t\right\} = 3/(s-2)$ which has denominator $Q(s) = s - 2$. Thus $p(s)Q(s) = (s-2)^2(s+3)$ and we compute the fundamental sets corresponding to $p(s)$ and $p(s)Q(s)$ as

   $$\begin{aligned} \mathrm{FS}_{p(s)Q(s)} &= \left\{e^{-3t},\, e^{2t},\, te^{2t}\right\} \\ \mathrm{FS}_{p(s)} &= \left\{e^{-3t},\, e^{2t}\right\}. \end{aligned}$$

   The only element in the first set that is not in the second is te^{2t}, so we conclude that a particular solution of the nonhomogeneous equation has the form $y_p = Ate^{2t}$ for some constant A to be computed by substitution in the differential equation. Since $y_p' = A(1+2t)e^{2t}$ and $y_p'' = A(4+4t)e^{2t}$, substitution of y_p in the differential equation gives

   $$3e^{2t} = y_p'' - 3y_p' - 4y_p = A(4+4t)e^{2t} + A(1+2t)e^{2t} - 6Ate^{2t} = 5Ae^{2t}.$$

   Hence, $5A = 3$, so $A = 3/5$ and $y_p = (3/5)te^{2t}$. The general solution is then

   $$\boxed{y = c_1e^{-3t} + c_2e^{2t} + \frac{3}{5}te^{2t}.}$$

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3. [**20 Points**] Find the unique solution of the initial value problem

   $$y'' + 2y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

   Is the equation under damped or over damped? Does $y(t) = 0$ for some $t > 0$? If so, find the first $t > 0$ for which $y(t) = 0$. Sketch the graph of your solution.

▶ **Solution.** The characteristic polynomial is $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$ which has roots $-1 \pm 2i$. Thus, the general solution of the homogeneous equation is

$$y = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t.$$

We need to choose c_1 and c_2 to satisfy the initial conditions. Since

$$y' = -c_1 e^{-t} \cos 2t - 2c_1 e^{-t} \sin 2t + 2c_2 e^{-t} \cos 2t - c_2 e^{-t} \sin 2t,$$

this means that c_1 and c_2 must satisfy the system of equations

$$\begin{aligned} 3 &= y(0) = c_1 \\ -1 &= y'(0) = -c_1 + 2c_2. \end{aligned}$$

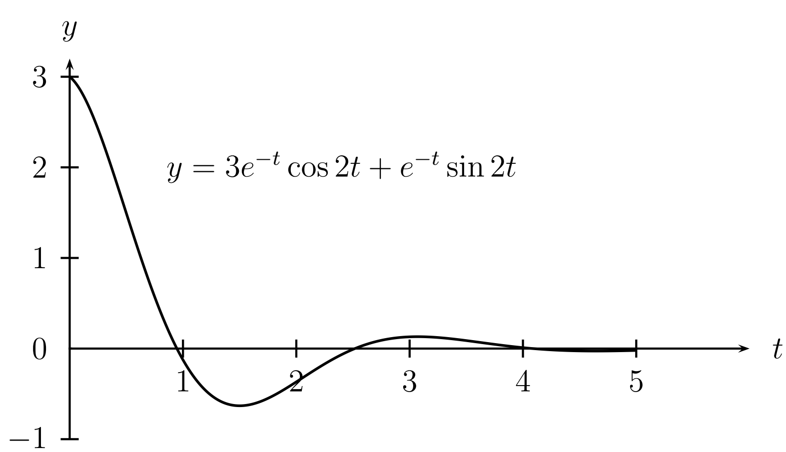
Hence, $c_1 = 3$ and $c_2 = 1$, so that the solution of the initial value problem is

$$\boxed{y = 3e^{-t} \cos 2t + e^{-t} \sin 2t.}$$

To answer the rest of the question, note that the characteristic polynomial has a pair of complex conjugate roots. Hence, the equation represents an **underdamped** system. To find the first $t > 0$ such that $y(t) = 0$, note that

$$\begin{aligned} y(t) = 0 \quad &\iff \quad 3\cos 2t + \sin 2t = 0 \\ &\iff \quad \tan 2t = -3 \\ &\iff \quad 2t \approx 1.8925 + \pi k \text{ for some integer } k \\ &\iff \quad t \approx .9462 + \frac{\pi}{2} k \text{ for some integer } k. \end{aligned}$$

The smallest such positive t is obtained for $k = 0$ and hence the first time that $y(t) = 0$ for $t > 0$ occurs for $t \approx .9462$.



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4. **[16 Points]** Find the Laplace transform of the following function:

$$f(t) = \begin{cases} \sin 2t & \text{if } 0 \le t < \pi, \\ 0 & \text{if } t \ge \pi. \end{cases}$$

▶ **Solution.** Write f in terms of the unit step functions as follows:

$$\begin{aligned} f(t) &= (\sin 2t)\chi_{[0,\pi)}(t) \\ &= (\sin 2t)(h(t) - h(t-\pi)) \\ &= \sin 2t - (\sin 2t)h(t-\pi). \end{aligned}$$

Then compute $F(s)$:

$$\begin{aligned} F(s) &= \frac{2}{s^2+4} - e^{-\pi s}\mathcal{L}\left\{\sin 2(t+\pi)\right\} \\ &= \frac{2}{s^2+4} - e^{-\pi s}\mathcal{L}\left\{\sin 2t\right\} \\ &= \frac{2}{s^2+4} - \frac{2}{s^+4}e^{-\pi s} \\ &= \frac{2}{s^2+4}(1 - e^{-\pi s}). \end{aligned}$$

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5. **[16 Points]** Find the inverse Laplace transform of the following function:

$$F(s) = \frac{(s+1)e^{-\pi s}}{s^2+2s+10}$$

▶ **Solution.** Note by completion of the square that $s^2+2s+10 = (s+1)^2+9$. Now let

$$G(s) = \frac{s+1}{s^2+2s+10} = \frac{s+1}{(s+1)^2+9},$$

so that (Formula C.2.10)

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = e^{-t}\cos 3t.$$

Then the second translation formula (Formula C.1.4) gives

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \mathcal{L}^{-1}\left\{G(s)e^{-\pi s}\right\} = g(t-\pi)h(t-\pi),$$

so that

$$\boxed{f(t) = (e^{-(t-\pi)}\cos 3(t-\pi))h(t-\pi).}$$

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6. **[16 Points]** Solve the following initial value problem:

$$y'' + 5y' + 6y = \delta(t-4), \quad y(0) = 0,\ y'(0) = 0.$$

Recall that $\delta(t-c)$ refers to the Dirac delta function which produces a unit impulse at time $t = c$. In the Table of Laplace transforms, this is referred to as $\delta_c(t)$ (that is $\delta_c(t) = \delta(t-c)$). (See Formula 24, Page 431 of the Laplace Transform Tables).

▶ **Solution.** Applying the Laplace transform to the differential equation, assuming that $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the solution function, gives

$$(s^2 + 5s + 6)Y(s) = \mathcal{L}\{\delta(t-4)\} = e^{-4s}.$$

Hence,

$$Y(s) = \frac{e^{-4s}}{s^2 + 5s + 6} = \frac{e^{-4s}}{(s+3)(s+2)} = \left(\frac{1}{s+2} - \frac{1}{s+3}\right)e^{-4s} = F(s)e^{-4s},$$

where

$$F(s) = \frac{1}{s+2} - \frac{1}{s+3}.$$

Since $f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-2t} - e^{-3t}$, we can apply Formula C.1.4 to get

$$\boxed{y(t) = f(t-4)h(t-4) = (e^{-2(t-4)} - e^{-3(t-4)})h(t-4).}$$

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