Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A short table of Laplace Transforms and a short table of integrals is included on Page 2.

1. [15 Points Each] Solve each of the following differential equations. If no initial condition is specified, give the general solution. If an initial condition is given, find the specific solution satisfying that initial condition. Be sure to show all of your work.

   (a) $y' = 2y + 4t$

   ▶ **Solution.** This equation is linear. Rewrite it in standard form as $y' - 2y = 4t$ so that an integrating factor is given by $\mu(t) = e^{-2t}$. Multiplying the equation by $\mu(t)$ gives $e^{-2t}y' - 2e^{-2t}y = 4te^{-2t}$. The left hand side is just $(e^{-2t}y)'$ so the equation becomes $(e^{-2t}y)' = 4te^{-2t}$ and integration gives:

$$\begin{aligned} e^{-2t}y &= \int 4te^{-2t}\,dt \\ &= -2te^{-2t} - e^{-2t} + C. \end{aligned}$$

   Multiplying by e^{2t} then gives $\boxed{y = -2t - 1 + Ce^{2t}}$. ◀

   (b) $y' = \dfrac{2t}{t^2y + y}$

   ▶ **Solution.** This equation can be written in the form $y' = \dfrac{2t}{t^2y+y} = \dfrac{2t}{y(t^2+1)}$, from which it is clear that it is separable. Multiplying by y gives the equation in separated variable form: $yy' = \dfrac{2t}{t^2+1}$. Write it is differential form as $y\,dy = \dfrac{2t}{t^2+1}\,dt$ and integrate to get

$$\frac{y^2}{2} = \ln(t^2+1) + C.$$

   Multiply by 2 to get

$$y^2 = 2\ln(t^2+1) + 2C = \ln(t^2+1)^2 + B$$

   where we have set $B = 2C$. Solving for y gives

$$\boxed{y = \pm\sqrt{\ln(t^2+1) + B}.}$$

   ◀

   (c) $2ty^2 + 1 + 2t^2yy' = 0$

▶ **Solution.** Letting $M(t, y) = 2ty^2 + 1$ and $N(t, y) = 2t^2y$, the equation can be written in the form $M + Ny' = 0$, and since

$$\frac{\partial M}{\partial y} = 4ty = \frac{\partial N}{\partial y},$$

it follows that the equation is exact. Hence, we solve it by looking for a potential function $V(t, y)$ such that $M = \partial V/\partial t$ and $N = \partial V/\partial y$. Find V by integration:

$$V(t, y) = \int \frac{\partial V}{\partial t}\, dt = \int M\, dt = \int (2ty^2 + 1)\, dt = t^2y^2 + t + \varphi(y),$$

where $\varphi(y)$ is a function depending only on y. Now use

$$2t^2y = N(t, y) = \frac{\partial V}{\partial y} = 2t^2y + \frac{d}{dy}\varphi(y),$$

which implies that $\frac{d}{dy}\varphi(y) = 0$ so that $\varphi(y)$ is a constant function. Hence, the solutions to the differential equation are given by the implicit solutions of $V(t, y) = C$, i.e.,

$$\boxed{t^2y^2 + t = C.}$$

Alternate Solution: Write the equation in the form $2ty^2 + 2t^2yy' = -1$ and notice that the left hand side is just the derivative (with respect to the variable t) of the function t^2y^2 so the equation becomes $(t^2y^2)' = -1$. Integration with respect to t then gives $t^2y^2 = -t + C$, so again we get that the solutions are the implicit solution of $t^2y^2 + t = C$. ◀

(d) $y' = y^2 - y, \quad y(0) = 1/3$

▶ **Solution.** Writing this equation in the form $y' + y = y^2$ shows that it is a Bernoulli equation. Dividing by y^2 gives the equation

$$y^{-2}y' + y^{-1} = 1.$$

Letting $z = y^{-1}$ and noting that $z' = -y^{-2}y'$ we see that the equation is transformed into the equation $-z' + z = 1$, or after multiplying by -1: $z' - z = -1$. This is a linear equation with an integrating factor $\mu(t) = e^{-t}$ so multiplying by e^{-t} gives $(e^{-t}z)' = -e^{-t}$, and integrating gives $e^{-t}z = -\int e^{-t}\, dt = e^{-t} + C$ so that $z = 1 + Ce^t$ and solving for y gives

$$y = \frac{1}{z} = \frac{1}{1 + Ce^t}.$$

The initial condition $y(0) = 1/3$ gives $1/3 = 1/(1 + C)$ so that $C = 2$. Hence,

$$\boxed{y = \frac{1}{1 + 2e^t}.}$$

Alternate Solution: This equation is also separable, so it can be solved using that technique also. Writing the equation is separated differential form gives $\dfrac{dy}{y^2 - y} = dt$, so the equation can be solved by integration:

$$\int \frac{dy}{y^2 - y} = \int dt = t + C,$$

and all that is needed is to compute the integral on the left hand side of the equation. But

$$\frac{1}{y^2 - y} = \frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y},$$

so that

$$t + C = \int \frac{dy}{y^2 - y} = \int \frac{dy}{y-1} - \int \frac{dy}{y} = \ln|(y-1)| - \ln|y| = \ln\left|\frac{y-1}{y}\right|.$$

Applying the exponential function gives

$$e^C e^t = \left|\frac{y-1}{y}\right|$$

or

$$\frac{y-1}{y} = Ke^t$$

where $K = \pm e^C$ is an arbitrary constant. Now solve for y: $y - 1 = yKe^t \implies y(1 - Ke^t) = 1$ so

$$y = \frac{1}{1 - Ke^t},$$

and as in the earlier solution, the initial condition $y(0) = 1/3$ gives $K = -2$, so that

$$y = \frac{1}{1 + 2e^t}.$$

◀

2. [10 Points] Apply Picard's method to compute the approximations $y_1(t)$, $y_2(t)$, and $y_3(t)$ to the solution of the initial value problem

$$y' = 2t + y, \quad y(0) = 0.$$

▶ **Solution.** This has the standard form $y' = F(t, y)$ where $F(t, y) = 2t + y$, and Picard's approximation starts with the constant approximation $y_0(t) = y_0 = 0$. Then

$$\begin{aligned} y_1(t) &= y_0 + \int_0^t F(u, y_0(u))\,du = 0 + \int_0^t 2u\,du = \boxed{t^2}; \\ y_2(t) &= \int_0^t F(u, y_1(u))\,du = \int_0^t (2u + u^2)\,du = \boxed{t^2 + \frac{y^3}{3}}; \\ y_3(t) &= \int_0^t F(u, y_2(u))\,du = \int_0^t (2u + u^2 + \frac{u^3}{3})\,du = \boxed{t^2 + \frac{y^3}{3} + \frac{t^4}{12}}. \end{aligned}$$

◀

3. [15 Points] Compute the Laplace transform of each of the following functions.

   (a) $f(t) = 3e^{-7t} - 7t^3$ $\boxed{F(s) = \dfrac{3}{s+7} - \dfrac{42}{s^4}}$

   (b) $g(t) = e^{-t/3}\cos\sqrt{6}t$ $\boxed{G(s) = \dfrac{s+\frac{1}{3}}{\left(s+\frac{1}{3}\right)^2 + 6}}$

   (c) $h(t) = t^2e^{3t} + t^3e^{2t}$ $\boxed{H(s) = \dfrac{2}{(s-3)^3} + \dfrac{6}{(s-2)^4}}$

4. [15 Points] A 1000 gallon tank is initially full of brine which contains 100 pounds of salt. A solution containing 4.0 pounds of salt per gallon enters the tank at a flow rate of 5 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same flow rate of 5 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time t.

   (a) What is $y(0)$? That is, how much salt is in the tank at time $t = 0$? $\boxed{y(0) = 100 \text{ lbs}}$

   (b) Find the amount $y(t)$ of salt in the tank for all times t.

   ▶ **Solution.** The balance equation is

   $$y'(t) = \text{rate in} - \text{rate out}.$$

   The rate in is 4.0 lb/gal × 5 gal/min, i.e., 20 lb/min. The rate out is

   $$\left(\frac{y(t)}{V(t)}\right) \times 5.$$

   Since mixture is entering and leaving at the same volume rate of 5 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 1000$. Hence $y(t)$ satisfies the equation

   $$y' = 20 - \frac{5}{1000}y,$$

   so that the initial value problem satisfied by $y(t)$ is

   $$y' + \frac{1}{200}y = 20, \qquad y(0) = 100.$$

   This is a linear differential equation with integrating factor $\mu(t) = e^{t/200}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

   $$\frac{d}{dt}\left(e^{t/200}y\right) = 20e^{t/200}.$$

   Integration of this equation gives

   $$e^{t/200}y = 4000e^{t/200} + C,$$

where C is an integration constant. Dividing by $e^{t/200}$ gives

$$y = 4000 + Ce^{-t/200},$$

and the initial condition $y(0) = 100$ gives a value of $C = -3900$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 4000 - 3900e^{-t/200}.}$$

◀

(c) How much salt is in the tank after 2 hours?

▶ **Solution.** This is obtained by taking $t = 120$ (minutes) in the previous equation:

$$\boxed{y(120) = 4000 - 3900e^{-120/200} \approx 1859.63 \text{ lb}}$$

◀

(d) What is $\lim_{t\to\infty} y(t)$? $\boxed{4000 \text{ lb}}$

Exam I Supplementary Sheet

A Short Table of Laplace Transforms

1.	$\mathcal{L}\{af(t)+bg(t)\}(s)$	$=$	$aF(s)+bG(s)$
2.	$\mathcal{L}\{e^{at}f(t)\}(s)$	$=$	$F(s-a)$
3.	$\mathcal{L}\{1\}(s)$	$=$	$\dfrac{1}{s}$
4.	$\mathcal{L}\{t^n\}(s)$	$=$	$\dfrac{n!}{s^{n+1}}$
5.	$\mathcal{L}\{e^{at}\}(s)$	$=$	$\dfrac{1}{s-a}$
6.	$\mathcal{L}\{t^n e^{\alpha t}\}(s)$	$=$	$\dfrac{n!}{(s-\alpha)^{n+1}}$
7.	$\mathcal{L}\{\cos bt\}(s)$	$=$	$\dfrac{s}{s^2+b^2}$
8.	$\mathcal{L}\{\sin bt\}(s)$	$=$	$\dfrac{b}{s^2+b^2}$
9.	$\mathcal{L}\{e^{at}\cos bt\}(s)$	$=$	$\dfrac{s-a}{(s-a)^2+b^2}$
10.	$\mathcal{L}\{e^{at}\sin bt\}(s)$	$=$	$\dfrac{b}{(s-a)^2+b^2}$

Some Integral Formulas

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (if $n \neq -1$)
2. $\int \dfrac{1}{x}\,dx = \ln|x| + C$
3. $\int \dfrac{1}{a+bx}\,dx = \dfrac{1}{b}\ln|a+bx| + C \quad (b \neq 0)$
4. $\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C \quad (a > 0)$
5. $\int \dfrac{1}{x(a+bx)}\,dx = \dfrac{1}{a}\ln\left|\dfrac{x}{a+bx}\right| + C$
6. $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C$
7. $\int \ln x\,dx = x\ln x - x + C$
8. $\int xe^{ax}\,dx = \dfrac{xe^{ax}}{a} - \dfrac{e^{ax}}{a^2} + C$