Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolution products, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

1. [**18 Points**] Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) in the table.

   (a) $f_1(t) = e^{2t}\cos 5t$

   ▶ **Solution.** From Formula 10, $\boxed{F_1(s) = \dfrac{s-2}{(s-2)^2+25}}$. ◀

   (b) $f_2(t) = 2t\cos 5t$

   ▶ **Solution.** Using Formulas 3 and 8:

$$\begin{aligned} F_2(s) &= -\frac{d}{ds}\mathcal{L}\{2\cos 5t\} \\ &= -\frac{d}{ds}\left(\frac{2s}{s^2+25}\right) \\ &= -\left(\frac{2(s^2+25)-2s(2s)}{(s^2+25)^2}\right) \\ &= \frac{2s^2-50}{(s^2+25)^2}. \end{aligned}$$

◀

   (c) $f_3(t) = e^{2t} * \cos 5t$ (Recall that $*$ denotes the convolution product.)

   ▶ **Solution.** Using Formulas 14, 6, and 8: $\boxed{F_3(s) = \dfrac{1}{s-2}\cdot\dfrac{2}{s^2+25}}$. ◀

2. [**22 Points**] Compute the inverse Laplace transform of each of the following rational functions.

   (a) $F(s) = \dfrac{3s+2}{s^2-2s-24}$

   ▶ **Solution.**

$$F(s) = \frac{3s+2}{s^2-2s-24} = \frac{3s+2}{(s-6)(s+4)} = \frac{A}{s-6} + \frac{B}{s+4}$$

where

$$A = \left.\frac{3s+2}{s+4}\right|_{s=6} = \frac{20}{10} = 2 \quad\text{and}\quad B = \left.\frac{3s+2}{s-6}\right|_{s=-4} = \frac{-10}{-10} = 1.$$

Hence, (Formula 6) $F(s) = \dfrac{2}{s-6} + \dfrac{1}{s+4}$ so $\boxed{f(t) = 2e^{6t} + e^{-4t}}$. ◀

(b) $G(s) = \dfrac{s-7}{s^2-2s+13}$

▶ **Solution.**

$$G(s) = \frac{s-7}{s^2-2s+13} = \frac{2-7}{(s-1)^2+12} = \frac{s-1}{(s-1)^2+12} - \frac{6}{(s-1)^2+12}.$$

Hence, (Formulas 10 and 11) $\boxed{g(t) = e^t \cos\sqrt{12}\,t - \dfrac{6}{\sqrt{12}} e^t \sin\sqrt{12}\,t}$. ◀

3. [**24 Points**] Find the *characteristic polynomial* and the general solution of each of the following constant coefficient linear homogeneous differential equations:

(a) $y'' + 8y' + 12y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2+8s+12 = (s+6)(s+2)$, which has roots -6 and -2. Thus, the general solution of the differential equation is $\boxed{y = c_1e^{-6t} + c_2e^{-2t}}$. ◀

(b) $y'' + 8y' + 16y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 8s + 16 = (s+4)^2$, which has a single root -4 of multiplicity 2. Thus, the general solution of the differential equation is $\boxed{y = c_1e^{-4t} + c_2te^{-4t}}$. ◀

(c) $y'' + 8y' + 20y = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 8s + 20 = (s+4)^2 + 4$, which has roots $-4 \pm 2i$. Thus, the general solution of the differential equation is $\boxed{y = c_1e^{-4t}\cos 2t + c_2e^{-4t}\sin 2t}$. ◀

(d) $y''' + 4y' = 0$

▶ **Solution.** The characteristic polynomial is $q(s) = s^3 + 4s = s(s^2+4)$, which has roots 0 and $\pm 2i$. Thus, the general solution of the differential equation is $\boxed{y = c_1 + c_2\cos 2t + c_3 \sin 2t}$. ◀

4. [**18 Points Each**] Use the Laplace transform method to find the solution of the following initial value problems:

(a) $y'' - 4y' + 3y = e^t, \qquad y(0) = 0,\ y'(0) = 4.$

▶ **Solution.** Let $y(t)$ be the unknown solution function and let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of $y(t)$. Taking the Laplace transform of both sides of the differential equation and using Formulas 12, 13, and 6, we get

$$s^2Y - 4 - 4sY + 3Y = \frac{1}{s-1}.$$

Rewrite this as

$$(s^2 - 4s + 3)Y = 4 + \frac{1}{s-1} = \frac{4s-3}{s-1},$$

and solve for Y to get

$$Y = \frac{4s-3}{(s-3)(s-1)^2}.$$

To compute $y(t) = \mathcal{L}^{-1}\{Y(s)\}$, expand Y in a partial fraction using the $(s-1)$-chain:

$$Y = \frac{4s-3}{(s-3)(s-1)^2} = \frac{A_1}{(s-1)^2} + \frac{p_1(s)}{(s-1)(s-3)},$$

where

$$A_1 = \left.\frac{4s-3}{s-3}\right|_{s=1} = -\frac{1}{2} \quad \text{and} \quad p_1(s) = \frac{(4s-3) + \frac{1}{2}(s-3)}{s-1} = \frac{\frac{9}{2}s - \frac{9}{2}}{s-1} = \frac{9}{2}.$$

Continuing the partial fraction calculation gives:

$$Y = \frac{-\frac{1}{2}}{(s-1)^2} + \frac{\frac{9}{2}}{(s-1)(s-3)} = \frac{-\frac{1}{2}}{(s-1)^2} - \frac{\frac{9}{4}}{s-1} + \frac{\frac{9}{4}}{s-3}.$$

Then $y(t) = \mathcal{L}^{-1}\{Y(s)\} = \boxed{-\frac{1}{2}te^t - \frac{9}{4}e^t + \frac{9}{4}e^{3t}}$. ◀

(b) $y'' + y = \cos t, \qquad y(0) = 0,\ y'(0) = 0.$

▶ **Solution.** Let $y(t)$ be the unknown solution function and let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of $y(t)$. Taking the Laplace transform of both sides of the differential equation and using Formulas 13 and 8, we get

$$s^2Y + Y = \frac{s}{s^2+1}.$$

Solve for Y to get

$$Y = \frac{s}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{s}{s^2+1}.$$

Then use Formulas 8, 9, and 14 to get

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} \cdot \frac{s}{s^2+1}\right\} = \sin t * \cos t = \frac{1}{2}t\sin t,$$

where the last equality comes from the convolution table (3rd formula from the bottom). ◀

Exam II Supplementary Sheets

A Short Table of Laplace Transforms

1. $\mathcal{L}\left\{af(t)+bg(t)\right\}(s) = aF(s)+bG(s)$

2. $\mathcal{L}\left\{e^{at}f(t)\right\}(s) = F(s-a)$

3. $\mathcal{L}\left\{-tf(t)\right\}(s) = \dfrac{d}{ds}F(s)$

4. $\mathcal{L}\left\{1\right\}(s) = \dfrac{1}{s}$

5. $\mathcal{L}\left\{t^n\right\}(s) = \dfrac{n!}{s^{n+1}}$

6. $\mathcal{L}\left\{e^{at}\right\}(s) = \dfrac{1}{s-a}$

7. $\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s) = \dfrac{n!}{(s-\alpha)^{n+1}}$

8. $\mathcal{L}\left\{\cos bt\right\}(s) = \dfrac{s}{s^2+b^2}$

9. $\mathcal{L}\left\{\sin bt\right\}(s) = \dfrac{b}{s^2+b^2}$

10. $\mathcal{L}\left\{e^{at}\cos bt\right\}(s) = \dfrac{s-a}{(s-a)^2+b^2}$

11. $\mathcal{L}\left\{e^{at}\sin bt\right\}(s) = \dfrac{b}{(s-a)^2+b^2}$

12. $\mathcal{L}\left\{f'(t)\right\}(s) = sF(s)-f(0)$

13. $\mathcal{L}\left\{f''(t)\right\}(s) = s^2F(s)-sf(0)-f'(0)$

14. $\mathcal{L}\left\{(f*g)(t)\right\}(s) = F(s)G(s)$

Table of Convolutions

$f(t)$	$g(t)$	$f * g(t)$
t	t^n	$\frac{t^{n+2}}{(n+1)(n+2)}$
t	$\sin at$	$\frac{at - \sin at}{a^2}$
t^2	$\sin at$	$\frac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
t	$\cos at$	$\frac{1 - \cos at}{a^2}$
t^2	$\cos at$	$\frac{2}{a^3}(at - \sin at)$
t	e^{at}	$\frac{e^{at} - (1 + at)}{a^2}$
t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
e^{at}	e^{at}	te^{at}
e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=\lambda} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1 s + C_1|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$