Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolution products, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

In Exercises 1 – 7, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' - \dfrac{2}{t}y = t^2 \cos t, \quad y(\pi/2) = 0.$
2. [**12 Points**] $y' = y^2 + ty^2, \quad y(0) = -4.$
3. [**12 Points**] $y'' + y' - 6y = 0, \quad y(0) = 1,\ y'(0) = 7.$
4. [**12 Points**] $4t^2y'' + 4ty' - 9y = 0.$
5. [**12 Points**] $y'' + 4y' + 29y = 0.$
6. [**12 Points**] $y'' + 8y' + 15y = 12e^{-3t}.$
7. [**12 Points**] $y'' + 16y = 8\delta_\pi(t), \quad y(0) = 3,\ y'(0) = -2.$ Recall that $\delta_c(t))$ refers to the Dirac delta function providing a unit impulse at time c.
8. [**12 Points**] Find a particular solution of the differential equation

$$t^2y'' - ty' - 8y = t^2,$$

given the fact that the general solution of the associated homogeneous equation is

$$y_h = c_1t^4 + c_2t^{-2}.$$

9. [**12 Points**] Find the Laplace transform of each of the following functions.

(a) $f(t) = \begin{cases} 0 & \text{if } 0 \le t < 2, \\ t^2 & \text{if } 2 \le t < 3 \\ 2 & \text{if } t \ge 3. \end{cases}$

(b) $g(t) = e^{2t}(t-1)^2 + e^{-2t}\sin 5t$

10. [**12 Points**] Compute each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{s-3}{s^2+2s+17}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{(s+2)(s^2+4)}\right\}$

11. [**18 Points**] Let $A = \begin{bmatrix} 3 & 5 \\ -2 & -3 \end{bmatrix}$.

    (a) Compute $(sI - A)^{-1}$.

    (b) Find $\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

    (c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

    (d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

12. [**12 Points**] A 200-gallon tank initially contains 150 gallons of pure water (i.e., no salt is initially dissolved in the water). Brine (a water-salt mixture) containing 2 pounds of salt per gallon flows into the tank at the rate of 3 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the same rate of 3 gal/min.

    (a) Find the amount of salt $y(t)$ in the tank at time t.

    (b) How much salt does the tank contain after 1 hour?

    (c) What is $\lim_{t\to\infty} y(t)$?

A Short Table of Laplace Transforms

1.	$\mathcal{L}\left\{af(t)+bg(t)\right\}(s)$	$=$	$aF(s)+bG(s)$
2.	$\mathcal{L}\left\{e^{at}f(t)\right\}(s)$	$=$	$F(s-a)$
3.	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
4.	$\mathcal{L}\left\{1\right\}(s)$	$=$	$\dfrac{1}{s}$
5.	$\mathcal{L}\left\{t^n\right\}(s)$	$=$	$\dfrac{n!}{s^{n+1}}$
6.	$\mathcal{L}\left\{e^{at}\right\}(s)$	$=$	$\dfrac{1}{s-a}$
7.	$\mathcal{L}\left\{t^n e^{\alpha t}\right\}(s)$	$=$	$\dfrac{n!}{(s-\alpha)^{n+1}}$
8.	$\mathcal{L}\left\{\cos bt\right\}(s)$	$=$	$\dfrac{s}{s^2+b^2}$
9.	$\mathcal{L}\left\{\sin bt\right\}(s)$	$=$	$\dfrac{b}{s^2+b^2}$
10.	$\mathcal{L}\left\{e^{at}\cos bt\right\}(s)$	$=$	$\dfrac{s-a}{(s-a)^2+b^2}$
11.	$\mathcal{L}\left\{e^{at}\sin bt\right\}(s)$	$=$	$\dfrac{b}{(s-a)^2+b^2}$
12.	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$sF(s)-f(0)$
13.	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2F(s)-sf(0)-f'(0)$
14.	$\mathcal{L}\left\{(f*g)(t)\right\}(s)$	$=$	$F(s)G(s)$
15.	$\mathcal{L}\left\{h(t-c)\right\}(s)$	$=$	$\dfrac{e^{-sc}}{s}$
16.	$\mathcal{L}\left\{f(t-c)h(t-c)\right\}(s)$	$=$	$e^{-sc}F(s)$
16'.	$\mathcal{L}\left\{g(t)h(t-c)\right\}(s)$	$=$	$e^{-sc}\mathcal{L}\left\{g(t+c)\right\}$
17.	$\mathcal{L}\left\{\delta_c(t)\right\}(s)$	$=$	e^{-sc}

Table of Convolutions

$f(t)$	$g(t)$	$f * g(t)$
t	t^n	$\dfrac{t^{n+2}}{(n+1)(n+2)}$
t	$\sin at$	$\dfrac{at - \sin at}{a^2}$
t^2	$\sin at$	$\dfrac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
t	$\cos at$	$\dfrac{1 - \cos at}{a^2}$
t^2	$\cos at$	$\dfrac{2}{a^3}(at - \sin at)$
t	e^{at}	$\dfrac{e^{at} - (1 + at)}{a^2}$
t^2	e^{at}	$\dfrac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
e^{at}	e^{bt}	$\dfrac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
e^{at}	e^{at}	te^{at}
e^{at}	$\sin bt$	$\dfrac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
e^{at}	$\cos bt$	$\dfrac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
$\sin at$	$\sin bt$	$\dfrac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at - at\cos at)$
$\sin at$	$\cos bt$	$\dfrac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$
$\cos at$	$\cos bt$	$\dfrac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at + \sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 0.1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \left.\frac{p_0(s)}{q(s)}\right|_{s=\lambda} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 0.2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1 s + C_1|_{s=a+bi} = \left.\frac{p_0(s)}{q(s)}\right|_{s=a+bi} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$