Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A table of Laplace transforms and a short table of integrals are appended to the exam.

1. [**17 Points**] Find the general solution of: $y' = 6t(y-1)^{2/3}$.

   ▶ **Solution.** This equation is separable. After separating variables, it becomes $(y-1)^{-2/3}y' = 6t$, which in differential form is $(y-1)^{-2/3}\,dy = 6t\,dt$ and integration then gives $3(y-1)^{1/3} = 3t^2 + C$. Dividing by 3 gives $(y-1)^{1/3} = t^2 + K$, where $K = C/3$ is an arbitrary constant. Cubing both sides and solving for y gives

   $$\boxed{y = (t^2+K)^3 + 1.}$$

   ◀

2. [**17 Points**] Find the general solution of: $y' - 4y = 3e^{4t} + 4e^{3t}$.

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = -4$ so that an integrating factor is given by $\mu(t) = e^{\int -4\,dt} = e^{-4t}$. Multiplication of the differential equation by the integrating factor gives

   $$e^{-4t}y' - 4e^{-4t}y = 3 + 4e^{-t},$$

   and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

   $$\frac{d}{dt}(e^{-4t}y) = 3 + 4e^{-t}.$$

   Integration then gives

   $$e^{-4t}y = 3t - 4e^{-t} + C,$$

   where C is an integration constant. Multiplying through by e^{4t} gives

   $$\boxed{y = 3te^{4t} - 4e^{3t} + Ce^{-4t}.}$$

   ◀

3. [**17 Points**] Solve the initial value problem: $y' + \dfrac{6}{t}y = 11t^4, \quad y(1) = 3.$

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = 6/t$, so that an integrating factor is $\mu(t) = e^{\int (6/t)\,dt} = 6e^{\ln t} = t^6$. Multiplication of the differential equation by the integrating factor gives

   $$t^6y' + 6t^5y = 11t^{10},$$

   and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

   $$\frac{d}{dt}(t^6y) = 11t^{10}.$$

Integration then gives

$$t^6 y = t^{11} + C,$$

where C is an integration constant. Multiplying through by t^{-6} gives

$$y = t^5 + Ct^{-6}.$$

The initial condition $y(1) = 3$ implies $3 = y(1) = 1 + C$ so that $C = 2$. Hence

$$\boxed{y = t^5 + 2t^{-6}.}$$

◀

4. [**17 Points**] Find the general solution of: $y \cos t + (\sin t + 3y^2)y' = 0$.

▶ **Solution.** Letting $M(t, y) = y \cos t$ and $N(t, y) = \sin t + 3y^2$, the equation can be written in the form $M + Ny' = 0$, and since

$$\frac{\partial M}{\partial y} = \cos t = \frac{\partial N}{\partial t},$$

it follows that the equation is exact. Hence, we solve it by looking for a potential function $V(t, y)$ such that $M = \partial V/\partial t$ and $N = \partial V/\partial y$. Find V by integration:

$$V(t, y) = \int \frac{\partial V}{\partial t}\, dt = \int M\, dt = \int y \cos t\, dt = y \sin t + \varphi(y),$$

where $\varphi(y)$ is a function depending only on y. Now use

$$\sin t + 3y^2 = N(t, y) = \frac{\partial V}{\partial y} = y \sin t + \frac{d}{dy}\varphi(y),$$

which implies that $\dfrac{d}{dy}\varphi(y) = 3y^2$ so that $\varphi(y) = y^3$. Hence, the solutions to the differential equation are given by the implicit solutions of $V(t, y) = C$, i.e.,

$$\boxed{y \sin + y^3 = C,}$$

where C is a constant. ◀

5. [**4 Points**] Complete the following definition: Suppose $f(t)$ is a continuous function defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows:

$$F(s) = \mathcal{L}\{f(t)\}(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

for all s sufficiently large.

6. **[12 Points]** Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) in the table.

   (a) $f(t) = 4t^3 - 5t^2 + 7$

   ▶ **Solution.** From Linearity and Formula 2, with $n = 3$, $n = 2$ and $n = 0$,

$$\boxed{F(s) = \frac{4 \cdot 3!}{s^4} + 5\frac{2}{s^3} + \frac{7}{s} = \frac{24}{s^4} + \frac{10}{s^3} + \frac{7}{s}.}$$

◀

   (b) $g(t) = 2e^{3t} + t^2e^{-2t} + 5e^t \cos 2t$

   ▶ **Solution.** From Linearity and Formulas 3, 4 and 7,

$$\boxed{G(s) = \frac{2}{s-3} + \frac{2}{(s+2)^3} + \frac{5(s-1)}{(s-1)^2+4}.}$$

◀

7. **[16 Points]** A tank contains 100 gallons of pure water. A solution containing 0.5 pounds of salt per gallon enters the tank at a flow rate of 3 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same flow rate of 3 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time t.

   (a) What is $y(0)$? That is, how much salt is in the tank at time $t = 0$? $\boxed{y(0) = 0 \text{ lbs}}$

   (b) Find the amount $y(t)$ of salt in the tank for all times t.

   ▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0.5 lb/gal × 3 gal/min, i.e., 1.5 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 3.$$

Since mixture is entering and leaving at the same volume rate of 3 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 100$. Hence $y(t)$ satisfies the equation

$$y' = 1.5 - \frac{3}{100}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$y' + \frac{3}{100}y = 1.5, \qquad y(0) = 0.$$

This is a linear differential equation with integrating factor $\mu(t) = e^{3t/100}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{3t/100}y\right) = 1.5e^{3t/100}.$$

Integration of this equation gives

$$e^{3t/100}y = 50e^{3t/100} + C,$$

where C is an integration constant. Dividing by $e^{3t/100}$ gives

$$y = 50 + Ce^{-3t/100},$$

and the initial condition $y(0) = 0$ gives a value of $C = -50$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 50 - 50e^{-3t/100}.}$$

◀

(c) How much salt is in the tank after 1 hour?

▶ **Solution.** This is obtained by taking $t = 60$ (minutes) in the previous equation:

$$\boxed{y(60) = 50 - 50e^{-180/100} \approx 41.74 \text{ lb}}$$

◀

(d) What is $\lim_{t\to\infty} y(t)$? $\boxed{50 \text{ lb}}$

Laplace Transform Tables

	$f(t)$	$\to$	$F(s) = \mathcal{L}\left\{f(t)\right\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$

Linearity	$\mathcal{L}\left\{af(t)+bg(t)\right\}$	$=$	$a\mathcal{L}\left\{f\right\}+b\mathcal{L}\left\{g\right\}$
Input Derivative Principles	$\mathcal{L}\left\{f'(t)\right\}(s)$	$=$	$s\mathcal{L}\left\{f(t)\right\}-f(0)$
	$\mathcal{L}\left\{f''(t)\right\}(s)$	$=$	$s^2\mathcal{L}\left\{f(t)\right\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\left\{e^{at}f(t)\right\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\left\{-tf(t)\right\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\left\{f(bt)\right\}(s)$	$=$	$\dfrac{1}{b}\mathcal{L}\left\{f(t)\right\}(s/b).$

Some Integral Formulas

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (if $n \neq -1$)
2. $\int \dfrac{1}{x}\,dx = \ln|x| + C$
3. $\int \dfrac{1}{a+bx}\,dx = \dfrac{1}{b}\ln|a+bx| + C \quad (b \neq 0)$
4. $\int \dfrac{1}{a^2+x^2}\,dx = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C \quad (a > 0)$
5. $\int \dfrac{1}{x(a+bx)}\,dx = \dfrac{1}{a}\ln\left|\dfrac{x}{a+bx}\right| + C$
6. $\int \dfrac{1}{a^2-x^2}\,dx = \dfrac{1}{2a}\ln\left|\dfrac{a+x}{a-x}\right| + C$
7. $\int \ln x\,dx = x\ln x - x + C$
8. $\int xe^{ax}\,dx = \dfrac{xe^{ax}}{a} - \dfrac{e^{ax}}{a^2} + C$
9. $\int \cos ax\,dx = \dfrac{\sin ax}{a} + C$
10. $\int \sin ax\,dx = -\dfrac{\cos ax}{a} + C$