Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned} \sin(\theta+\varphi) &= \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi \end{aligned}}$$

1. [**10 Points**] Find the solution of the following initial value problem:

$$t^2y'' + 2ty' - 12y = 0, \quad y(1) = 0,\ y'(1) = 1.$$

2. [**15 Points**] Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

$$y'' + 4y' + 4y = t^{-5}e^{-2t}.$$

You may assume that the solution of the homogeneous equation $y'' + 4y' + 4y = 0$ is $y_h = c_1e^{-2t} + c_2te^{-2t}$.

3. [**15 Points**] Find the Laplace transform of the following function:

$$f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi, \\ -2 & \text{if } t \ge \pi. \end{cases}$$

4. [**20 Points**] Find the inverse Laplace transform of the following functions:

   (a) $F(s) = \dfrac{1}{(s-2)}e^{-3s} + \dfrac{1}{(s-2)^2}e^{-4s}$

   (b) $G(s) = \dfrac{s}{s^2+4}e^{-\frac{\pi}{2}s}$

5. [**15 Points**] Solve the following initial value problem:

$$y'' + 4y = 16h(t-\pi), \qquad y(0) = 0,\ y'(0) = 1.$$

6. [**25 Points**] Let $A = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}$.

   (a) Compute $(sI - A)^{-1}$.

   (b) Find $\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

   (c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

   (d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\frac{1}{s}$
2.	t^n	$\to$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\frac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\frac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\frac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\{f(bt)\}(s)$	$=$	$\frac{1}{b}\mathcal{L}\{f(t)\}(s/b).$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s).$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (**Linear Case**). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (**Irreducible Quadratic Case**). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where $s^2 + cs + d$ *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$