Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned} \sin(\theta+\varphi) &= \sin\theta\cos\varphi+\sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi-\sin\theta\sin\varphi \end{aligned}}$$

1. [**10 Points**] Find the solution of the following initial value problem:

$$t^2y''+2ty'-12y=0, \quad y(1)=0,\ y'(1)=1.$$

▶ **Solution.** This is a Cauchy-Euler equation with indicial polynomial

$$q(s)=s(s-1)+2s-12=(s+4)(s-3).$$

This $q(s)$ has roots -4 and 3 so the general solution is $y(t)=c_1t^{-4}+c_2t^3$. Plugging in the initial conditions gives

$$\begin{aligned} 0=y(1) &= c_1+c_2 \\ 1=y'(1) &= -4c_1+3c_2. \end{aligned}$$

Solving for c_1 and c_2 gives $c_1=-1/7$ and $c_2=1/7$. Thus,

$$\boxed{y(t)=-\frac{1}{7}t^{-4}+\frac{1}{7}t^3.}$$

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2. [**15 Points**] Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

$$y''+4y'+4y=t^{-5}e^{-2t}.$$

You may assume that the solution of the homogeneous equation $y''+4y'+4y=0$ is $y_h=c_1e^{-2t}+c_2te^{-2t}$.

▶ **Solution.** Letting $y_1=e^{-2t}$ and $y_2=te^{-2t}$, a particular solution has the form

$$y_p=u_1y_1+u_2y_2=u_1e^{-2t}+u_2te^{-2t},$$

where u_1 and u_2 are unknown functions whose derivatives satisfy the simultaneous equations

$$\begin{aligned} u_1'e^{-2t}+u_2'te^{-2t} &= 0 \\ -2u_1'e^{-2t}+u_2'(e^{-2t}-2te^{-2t}) &= t^{-5}e^{-2t}. \end{aligned}$$

Multiplying both equations by e^{2t} gives

$$\begin{aligned} u_1' + u_2't &= 0 \\ -2u_1' + u_2'(1-2t) &= t^{-5}. \end{aligned}$$

Eliminating u_1' gives $u_2' = t^{-5}$ and then $u_1' = -tu_2' = -t^{-4}$. Integrating gives $u_1 = (1/3)t^{-3}$ and $u_2 = (-1/4)t^{-4}$ so that

$$y_p = \frac{1}{3}t^{-3}e^{-2t} - \frac{1}{4}t^{-4}te^{-2t}.$$

Hence

$$\boxed{y_p = \frac{1}{12}t^{-3}e^{-2t}.}$$

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3. **[15 Points]** Find the Laplace transform of the following function:

$$f(t) = \begin{cases} \cos t & \text{if } 0 \le t < \pi, \\ -2 & \text{if } t \ge \pi. \end{cases}$$

▶ **Solution.** Use characteristic functions to write $f(t)$ in terms of unit step functions:

$$\begin{aligned} f(t) &= (\cos t)\chi_{[0,\,\pi)}(t) - 2\chi_{[\pi,\,\infty)}(t) \\ &= (\cos t)(h(t) - h(t-\pi)) - 2h(t-\pi) \\ &= \cos t - (\cos t + 2)h(t-\pi). \end{aligned}$$

Now apply the second translation theorem to get

$$\begin{aligned} F(s) = \mathcal{L}\left\{f(t)\right\} &= \frac{s}{s^2+1} - e^{-\pi s}\mathcal{L}\left\{\cos(t+\pi) + 2\right\} \\ &= \frac{s}{s^2+1} + e^{\pi s}\frac{s}{s^2+1} - \frac{2}{s}e^{-\pi s}. \end{aligned}$$

The + sign in the second term results from the fact that

$$\cos(t+\pi) = \cos t\cos(\pi) - \sin t \sin \pi = -\cos t.$$

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4. **[20 Points]** Find the inverse Laplace transform of the following functions:

   (a) $F(s) = \dfrac{1}{(s-2)}e^{-3s} + \dfrac{1}{(s-2)^2}e^{-4s}$

   ▶ **Solution.** $\boxed{f(t) = e^{2(t-3)}h(t-3) + (t-4)e^{2(t-4)}h(t-4)}$ ◀

(b) $G(s) = \dfrac{s}{s^2+4}e^{-\frac{\pi}{2}s}$

▶ **Solution.** $\boxed{g(t) = (\cos 2(t-\pi/2))h(t-\pi/2) = -(\cos 2t)h(t-\pi/2)}$ ◀

5. [**15 Points**] Solve the following initial value problem:

$$y'' + 4y = 16h(t-\pi), \qquad y(0) = 0,\ y'(0) = 1.$$

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the solution of the differential equation and apply the Laplace transform to both sides of the equation to get

$$s^2Y(s) - 1 + 4Y(s) = \frac{16}{s}e^{-\pi s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{1}{s^2+4} + \frac{16}{s(s^2+4)}e^{-\pi s}.$$

Use partial fractions to write

$$\frac{16}{s(s^2+4)} = \frac{4}{s} - \frac{4s}{s^2+4},$$

which then gives

$$Y(s) = \frac{1}{s^2+4} + \left(\frac{4}{s} - \frac{4s}{s^2+4}\right)e^{-\pi s}.$$

Taking inverse Laplace transforms gives

$$\begin{aligned} y(t) &= \frac{1}{2}\sin 2t + (4 - 4\cos 2(t-\pi))h(t-\pi) \\ &= \frac{1}{2}\sin 2t + (4 - 4\cos 2t)h(t-\pi) \\ &= \begin{cases} \frac{1}{2}\sin 2t, & \text{if } 0 \le t < \pi; \\ \frac{1}{2}\sin 2t + 4 - 4\cos 2t, & \text{if } t \ge \pi. \end{cases} \end{aligned}$$

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6. [**25 Points**] Let $A = \begin{bmatrix} 1 & 5 \\ -2 & -1 \end{bmatrix}$.

(a) Compute $(sI - A)^{-1}$.

▶ **Solution.** $sI - A = \begin{bmatrix} s-1 & -5 \\ 2 & s+1 \end{bmatrix}$ so $\det(sI-A) = (s-1)(s+1)+10 = s^2+9/$ Hence,

$$(sI-A)^{-1} = \frac{1}{s^2+9}\begin{bmatrix} s+1 & 5 \\ -2 & s-1 \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s^2+9} & \dfrac{5}{s^2+9} \\ \dfrac{-2}{s^2+9} & \dfrac{s-1}{s^2+9} \end{bmatrix}.$$

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(b) Find $\mathcal{L}^{-1}\{(sI-A)^{-1}\}$.

▶ **Solution.**

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} \cos 3t + \frac{1}{3}\sin 3t & \frac{5}{3}\sin 3t \\ -\frac{2}{3}\sin 3t & \cos 3t - \frac{1}{3}\sin 3t \end{bmatrix}.$$

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(c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.** The general solution is $\mathcal{L}^{-1}\{(sI-A)^{-1}\}\,\mathbf{y}(0)$. Thus, if $\mathbf{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then the general solution is

$$\mathbf{y}(t) = \begin{bmatrix} c_1\cos 3t + \left(\frac{1}{3}c_1 + \frac{5}{3}c_2\right)\sin 3t \\ c_2\cos 3t - \left(\frac{2}{3}c_1 + \frac{1}{3}c_2\right)\sin 3t \end{bmatrix}$$

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(d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

▶ **Solution.** Setting $c_1 = 2$, $c_2 = -1$ in part (c) gives

$$\mathbf{y}(t) = \begin{bmatrix} 2\cos 3t - \sin 3t \\ -\cos 3t - \sin 3t \end{bmatrix}.$$

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Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\{f(bt)\}(s)$	$=$	$\dfrac{1}{b}\mathcal{L}\{f(t)\}(s/b).$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s).$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. Then there is a unique number A_1 and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 and the polynomial $p_1(s)$ are given by

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \textit{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d is an irreducible quadratic that is factored completely out of $q(s)$. Then there is a unique linear term $B_1 s + C_1$ and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ is a complex root of s^2+cs+d then $B_1 s + C_1$ and the polynomial $p_1(s)$ are given by

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \textit{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$