Math 2065 Section 1
Review Exercises for Exam I
Answers

1. (a) L
   (b) S, E
   (c) B
   (d) N
   (e) S, L, E
   (f) H, E
   (g) E
   (h) L

2. (a) $y(t) = ce^{-2t}$
   (b) $y(t) = ce^{-2t} + e^t$
   (c) $y(t) = ce^t + \frac{1}{2}e^{3t}$
   (d) $y(t) = ce^{-t^2} + \frac{1}{2}$
   (e) $y(t) = ct^{-3} - t^{-3}\cos t$
   (f) $y^2 - t^2 = c$
   (g) $y(t) = \sin(\ln|t| + c)$
   (h) $y(t) = 2\tan(2t + c)$
   (i) $t^2y^2 - 3t + 4y = c$
   (j) $ty - 2te^t + 2e^t - 2t^3 = c$
   (k) $y = e^{-t}/(c - t)$

3. (a) $y(t) = 5te^{2t} - e^{2t}$
   (b) $y(t) = -4/(t^4 + 3)$
   (c) $y(t) = -2e^{-3t}\cos 2t + e^{-3t}$
   (d) $y(t) = t^4 - \frac{2}{t^3}$
   (e) $4ty + t^2 - 5t + 3y^2 - y = 8$

4. $T' = k(T - 350)$, $T(0) = 70$ (k is a proportionality constant)

5. (a) $y(t) = 600 - 550e^{-0.01t}$ (b) 600 lbs. This makes sense because, after a long period of time, the amount of salt in the tank should be close to the "steady state" value which corresponds to the concentration of the solution flowing into the tank. That is, 300 gal times 2 lb/gal = 600 lbs of salt.

6. $y_1(t) = \frac{t^3}{3}, \quad y_2(t) = \frac{t^3}{3} + \frac{t^7}{63}$

7. (a) Complete the following definition: Suppose $f(t)$ is a continuous function of exponential type defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows

$$F(s) = \mathcal{L}(f(t))(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

for all s sufficiently large.

(b) Using your definition compute the Laplace transform of the function $f(t) = 2t - 5$. You may need to review the integration by parts formula: $\int u\, dv = uv - \int v\, du$.

▶ **Solution.** The Laplace transform of $f(t) = 2t - 5$ is the integral

$$\mathcal{L}(2t-5)(s) = \int_0^\infty (2t-5)e^{-st}\, dt,$$

which is computed using the integration by parts formula by letting $u = 2t - 5$ and $dv = e^{-wt}\, dt$, so that $du = 2\, dt$ while $v = -\frac{1}{s}e^{-st}$. Thus, if $s > 0$,

$$\begin{aligned}
\mathcal{L}(2t-5)(s) &= \int_0^\infty (2t-5)e^{-st}\, dt \\
&= -\frac{2t-5}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{2}{s}e^{-st}\, dt \\
&= \left(-\frac{2t-5}{s}e^{-st} - \frac{2}{s^2}e^{-st}\right)\Big|_0^\infty \\
&= -\frac{5}{s} + \frac{2}{s^2}.
\end{aligned}$$

The last evaluation uses the fact (verified in calculus) that $\lim_{t\to\infty} e^{-st} = 0$ and $\lim_{t\to\infty} te^{-st} = 0$ provided $s > 0$. ◀

8. Compute the Laplace transform of each of the following functions using the Laplace transform Tables (See Pages 103–104). (Laplace Transform tables will be provided to you on the exam.)

(a) $f(t) = 3t^3 - 2t^2 + 7$

$$\boxed{F(s) = 3\frac{3!}{s^4} - 2\frac{2!}{s^3} + \frac{7}{s} = \frac{18}{s^4} - \frac{4}{s^3} + \frac{7}{s}.}$$

(b) $g(t) = e^{-3t} + \sin\sqrt{2}t$

$$\boxed{G(s) = \frac{1}{s+3} + \frac{\sqrt{2}}{s^2+2}.}$$

(c) $h(t) = -8 + \cos(t/2)$

$$\boxed{H(s) = -\frac{8}{s} + \frac{2}{s^2+1/4} = -\frac{8}{s} + \frac{4s}{4s^2+1}.}$$

(d) $f(t) = 7e^{2t}\cos 3t - 2e^{7t}\sin 5t$

$$\boxed{F(s) = \frac{7s}{(s-2)^2+9} - \frac{10}{(s-7)^2+25}.}$$

(e) $g(t) = 3t\sin 2t$

▶ **Solution.** Use the Transform Derivative Principle, Page 100: $\mathcal{L}\{tf(t)\}(s) = -F'(s)$. Apply this formula to the function $f(t) = 3\sin 2t$ so that $F(s) = 6/(s^2+4)$. Since $g(t) = tf(t)$, formula 21) gives:

$$G(s) = -F'(s) = -\frac{-12s}{(s^2+4)^2} = \frac{12s}{(s^2+4)^2}.$$

◀

(f) $h(t) = (2 - t^2)e^{-5t}$

▶ **Solution.** Use linearity and formula 15, Page 99. Then

$$\boxed{H(s) = \frac{2}{s+5} - \frac{2}{(s+5)^3}.}$$

◀

(g) t^2e^{-9t}

$$\boxed{\frac{2}{(s+9)^3}}$$

(h) $e^{2t} - t^3 + t^2 - \sin 5t$

$$\boxed{\frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}}$$

(i) $t\cos 6t$

$$\boxed{-\frac{d}{ds}\left(\frac{s}{s^2+36}\right) = \frac{s^2-36}{(s^2+36)^2}}$$

(j) $2\sin t + 3\cos 2t$

$$\boxed{\frac{2}{s^2+1} + \frac{3s}{s^2+4}}$$

(k) $e^{-5t}\sin 6t$

$$\boxed{\frac{6}{(s+5)^2+36}}$$

(l) $t^2 \cos at$ where a is a constant

► **Solution.** Use the Transform n^{th}-Derivative Principle, Page 101, applied to $f(t) = \cos at$. Then, $F(s) = s/(s^2+a^2)$ and $\mathcal{L}\{t^2\cos at\}(s) = F''(s)$. Since $F'(s) = (a^2-s^2)/(s^2+a^2)^2$, the Laplace transform of $t^2\cos at$ is

$$\boxed{F''(s) = \frac{2s^2 - 6sa^2}{(s^2+a^2)^3}.}$$

◄

(m) $f(t) = 3(e^t)^4 + \sin\sqrt{2}t$ $\quad \boxed{F(s) = \frac{3}{s-4} + \frac{\sqrt{2}}{s^2+2}}$

(n) $g(t) = 5t^3 - 3\cos 5t + \frac{3}{5}$ $\quad \boxed{G(s) = \frac{30}{s^4} - \frac{3s}{s^2+25} + \frac{3}{5s}}$