Math 2065
Review Exercises for Exam II

The syllabus for Exam II is Sections 2.2 – 2.6 of Chapter 2 and Sections 3.1 – 3.4 of Chapter 3. You should review all of the assigned exercises in these sections. In addition, Section 2.1 contains the main formulas for computation of Laplace transforms, and hence, while not explicitly part of the exam syllabus, it will be necessary to be completely conversant with the computational techniques of Section 2.1. For this reason, some of the exercises related to this section from the last review sheet are repeated here. A Laplace transform table will be provided with the test. Following is a brief list of terms, skills, and formulas with which you should be familiar.

- Know how to use all of the Laplace transform formulas developed in Section 2.1 to be able to compute the Laplace transform of elementary functions.

- Know how to use partial fraction decompositions to be able to compute the inverse Laplace transform of any proper rational function. The key recursion algorithms for computing partial fraction decompositions are Theorem 1 (Page 108) for the case of a real root in the denominator, and Theorem 1 (Page 117) for a complex root in the denominator. Here are the two results:

  Theorem 1 (Linear Partial Fraction Recursion). *Let $P_0(s)$ and $Q(s)$ be polynomials. Assume that a is a number such that $Q(a) \neq 0$ and n is a positive integer. Then there is a unique number A_1 and polynomial $P_1(s)$ such that*

  $$\frac{P_0(s)}{(s-a)^n Q(s)} = \frac{A_1}{(s-a)^n} + \frac{P_1(s)}{(s-a)^{n-1}Q(s)}.$$

  The number A_1 and polynomial $P_1(s)$ are computed as follows:

  $$A_1 = \left.\frac{P_0(s)}{Q(s)}\right|_{s=a} \quad \textit{and} \quad P_1(s) = \frac{P_0(s) - A_1 Q(s)}{s-a}.$$

  Theorem 2 (Quadratic Partial Fraction Recursion). *Let $P_0(s)$ and $Q(s)$ be polynomials. Assume that $a + bi$ is a complex number with nonzero imaginary part $b \neq 0$ such that $Q(a+bi) \neq 0$ and n is a positive integer. Then there is a unique linear term $B_1 s + C_1$ and polynomial $P_1(s)$ such that*

  $$\frac{P_0(s)}{((s-a)^2+b^2)^n Q(s)} = \frac{B_1 s + C_1}{((s-a)^2+b^2)^n} + \frac{P_1(s)}{((s-a)^2+b^2)^{n-1}Q(s)}.$$

  The linear term $B_1 s + C_1$ and polynomial $P_1(s)$ are computed as follows:

  $$\left. B_1 s + C_1\right|_{s=a+bi} = \left.\frac{P_0(s)}{Q(s)}\right|_{s=a+bi} \quad \textit{and} \quad P_1(s) = \frac{P_0(s) - (B_1 s + C_1)Q(s)}{(s-a)^2+b^2}.$$

- Know how to apply the input derivative principle (Corollary 7, Page 95) to compute the solution of an initial value problem for a constant coefficient linear differential equation with elementary forcing function. See Section 3.1.

- Know how to use the *characteristic polynomial* to be able to solve constant coefficient homogeneous linear differential equations. (See Algorithm 4, Page 186.)

- Know how to use the *method of undetermined coefficients* to find a particular solution of the constant coefficient linear differential equation

$$ay'' + by' + cy = f(t)$$

where the forcing function $f(t)$ is an exponential polynomial, i.e., $f(t)$ is a sum of functions of the form $ct^k e^{at}\cos bt$ and $dt^k e^{at}\sin bt$ for various choices of the constants a, b, c, and d, and non-negative integer k. (See Algorithm 4, Page 190.)

The following is a small set of exercises of types identical to those already assigned.

1. (a) Complete the following definition: Suppose $f(t)$ is a continuous function of exponential type defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows

$$F(s) = \mathcal{L}(f(t))(s) = \boxed{\quad \int_0^\infty e^{-st} f(t)\, dt \quad}$$

for all s sufficiently large.

(b) Using your definition compute the Laplace transform of the function $f(t) = 2t - 5$. You may need to review the integration by parts formula: $\int u\, dv = uv - \int v\, du$.

▶ **Solution.** The Laplace transform of $f(t) = 2t - 5$ is the integral

$$\mathcal{L}(2t-5)(s) = \int_0^\infty (2t-5)e^{-st}\, dt,$$

which is computed using the integration by parts formula by letting $u = 2t - 5$ and $dv = e^{-wt}\, dt$, so that $du = 2\, dt$ while $v = -\frac{1}{s}e^{-st}$. Thus, if $s > 0$,

$$\begin{aligned}
\mathcal{L}(2t-5)(s) &= \int_0^\infty (2t-5)e^{-st}\, dt \\
&= -\frac{2t-5}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{2}{s}e^{-st}\, dt \\
&= \left(-\frac{2t-5}{s}e^{-st} - \frac{2}{s^2}e^{-st}\right)\Big|_0^\infty \\
&= -\frac{5}{s} + \frac{2}{s^2}.
\end{aligned}$$

The last evaluation uses the fact (verified in calculus) that $\lim_{t\to\infty} e^{-st} = 0$ and $\lim_{t\to\infty} te^{-st} = 0$ provided $s > 0$. ◀

2. Compute the Laplace transform of each of the following functions using the Laplace transform Tables (See Pages 103–104). (A table of Laplace Transforms will be provided to you on the exam.)

   (a) $f(t) = 3t^3 - 2t^2 + 7$

$$\boxed{F(s) = 3\frac{3!}{s^4} - 2\frac{2!}{s^3} + \frac{7}{s} = \frac{18}{s^4} - \frac{4}{s^3} + \frac{7}{s}.}$$

   (b) $g(t) = e^{-3t} + \sin\sqrt{2}t$

$$\boxed{G(s) = \frac{1}{s+3} + \frac{\sqrt{2}}{s^2+2}.}$$

   (c) $h(t) = -8 + \cos(t/2)$

$$\boxed{H(s) = -\frac{8}{s} + \frac{2}{s^2+1/4} = -\frac{8}{s} + \frac{4s}{4s^2+1}.}$$

3. Compute the Laplace transform of each of the following functions. You may use the Laplace Transform Tables.

   (a) $f(t) = 7e^{2t}\cos 3t - 2e^{7t}\sin 5t$

$$\boxed{F(s) = \frac{7s}{(s-2)^2+9} - \frac{10}{(s-7)^2+25}.}$$

   (b) $g(t) = 3t\sin 2t$

   ▶ **Solution.** Use the transform derivative principle: $\mathcal{L}\{tf(t)\}(s) = -F'(s)$. Apply this formula to the function $f(t) = 3\sin 2t$ so that $F(s) = 6/(s^2+4)$. Since $g(t) = tf(t)$, the transform derivative princple gives:

$$G(s) = -F'(s) = -\frac{-12s}{(s^2+4)^2} = \frac{12s}{(s^2+4)^2}.$$

◀

   (c) $h(t) = (2 - t^2)e^{-5t}$

   ▶ **Solution.** Use the first translation principle. Then

$$\boxed{H(s) = \frac{2}{s+5} - \frac{2}{(s+5)^3}.}$$

◀

4. Find the inverse Laplace transform of each of the following functions. You may use the Laplace Transform Tables.

(a) $F(s) = \dfrac{7}{(s+3)^3}$

$$\boxed{f(t) = \frac{7}{2}t^2e^{-3t}.}$$

(b) $G(s) = \dfrac{s+2}{s^2-3s-4}$

▶ **Solution.** Use partial fractions to write

$$G(s) = \frac{s+2}{s^2-3s-4} = \frac{1}{5}\left(\frac{6}{s-4} - \frac{1}{s+1}\right).$$

Thus $\boxed{g(t) = \left(6e^{4t} - e^{-t}\right)/5}$. ◀

(c) $H(s) = \dfrac{s}{(s+4)^2+4}$

▶ **Solution.** Since

$$H(s) = \frac{s}{(s+4)^2+4} = \frac{(s+4)-4}{(s+4)^2+4} = \frac{s+4}{(s+4)^2+4} - 2\frac{2}{(s+4)^2+4},$$

it follows that $\boxed{h(t) = e^{-4t}\cos 2t - 2e^{-4t}\sin 2t}$. ◀

5. Find the Laplace transform of each of the following functions.

(a) t^2e^{-9t}

$$\boxed{\frac{2}{(s+9)^3}}$$

(b) $e^{2t} - t^3 + t^2 - \sin 5t$

$$\boxed{\frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}}$$

(c) $t\cos 6t$

$$\boxed{-\frac{d}{ds}\left(\frac{s}{s^2+36}\right) = \frac{s^2-36}{(s^2+36)^2}}$$

(d) $2\sin t + 3\cos 2t$

$$\boxed{\frac{2}{s^2+1} + \frac{3s}{s^2+4}}$$

(e) $e^{-5t}\sin 6t$

$$\boxed{\frac{6}{(s+5)^2+36}}$$

(f) $t^2\cos at$ where a is a constant

▶ **Solution.** Use the transform derivative principal twice, applied to $f(t)=\cos at$. Then, $F(s)=s/(s^2+a^2)$ and $\mathcal{L}\left\{t^2\cos at\right\}(s)=F''(s)$. Since $F'(s)=(a^2-s^2)/(s^2+a^2)^2$, the Laplace transform of $t^2\cos at$ is

$$\boxed{F''(s)=\frac{2s^2-6sa^2}{(s^2+a^2)^3}.}$$

◀

6. Find the inverse Laplace transform of each of the following functions.

(a) $\dfrac{1}{s^2-10s+9}$

▶ **Solution.** Since $s^2-10s-9=(s-9)(s-1)$, use partial fractions:

$$\frac{1}{s^2-10s+9}=\frac{1}{8}\left(\frac{1}{s-9}-\frac{1}{s-1}\right)\Rightarrow\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2-10s+9}\right\}=\frac{1}{8}(e^{9t}-e^{t}).}$$

◀

(b) $\dfrac{2s-18}{s^2+9}$ $\boxed{2\cos 3t-6\sin 3t}$

(c) $\dfrac{2s+18}{s^2+25}$ $\boxed{2\cos 5t+(18/5)\sin 5t}$

(d) $\dfrac{s+3}{s^2+5}$ $\boxed{\cos\sqrt{5}t+(3/\sqrt{5})\sin\sqrt{5}t}$

(e) $\dfrac{s-3}{s^2-6s+25}$

▶ **Solution.** Since $s^2-6s+25=(s-3)^2+4^2$, we conclude:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-3}{s^2-6s+25}\right\}=e^{3t}\cos 4t.}$$

◀

(f) $\dfrac{1}{s(s^2+4)}$

▶ **Solution.** Use partial fractions to write

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right),$$

so that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}(1-\cos 2t)\,.}$$

◀

(g) $\dfrac{1}{s^2(s+1)^2}$

▶ **Solution.** Use partial fractions to write

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2} - \frac{2}{s} + \frac{1}{(s+1)^2} + \frac{2}{s+1},$$

so that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = te^{-t} + 2e^{-t} + t - 2.}$$

◀

7. Compute the convolution $t * t^3$ directly from the definition.

▶ **Solution.**

$$\begin{aligned}
t * t^3 &= \int_0^t \tau(t-\tau)^3\,d\tau \\
&= \int_0^t \tau(t^3 - 3t^2\tau + 3t\tau^2 - \tau^3)\,d\tau \\
&= \int_0^t \left(t^3\tau - 3t^2\tau^2 + 3t\tau^3 - \tau^4\right)\,d\tau \\
&= \left(t^3\frac{\tau^2}{2} - t^2\tau^3 + \frac{3}{4}t\tau^4 - \frac{\tau^5}{5}\right)\Bigg|_0^t \\
&= t^5\left(\frac{1}{2} - 1 + \frac{3}{4} - \frac{1}{5}\right) \\
&= \frac{t^5}{20}.
\end{aligned}$$

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8. Using the table of convolutions (Page 167), compute each of the following convolutions:

(a) $(1+3t) * e^{5t}$

▶ **Solution.**

$$\begin{aligned}(1+3t) * e^{5t} &= 1 * e^{5t} + 3t * e^{5t} \\ &= \int_0^t e^{5\tau}\, d\tau + 3\left(\frac{e^{5t} - (1+5t)}{25}\right) \\ &= \frac{1}{5}(e^{5t} - 1) + 3\left(\frac{e^{5t} - (1+5t)}{25}\right) \\ &= \frac{8e^{5t} - 8 - 15t}{25}.\end{aligned}$$

◀

(b) $(1/2 + 2t^2) * \cos\sqrt{2}t$ $\boxed{2t - \frac{3\sqrt{2}}{4}\sin\sqrt{2}t}$

(c) $(e^{2t} - 3e^{4t}) * (e^{2t} + 4e^{3t})$ $\boxed{te^{2t} - \frac{5}{2}e^{2t} + 16e^{3t} - \frac{27}{2}e^{4t}}$

9. Solve each of the following differential equations by means of the Laplace transform:

(a) $y'' - 3y' + 2y = 4, \qquad y(0) = 2, \quad y'(0) = 3$

▶ **Solution.** As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation gives

$$s^2Y(s) - 2s - 3 - 3(Y(s) - 2) + 2Y(s) = \frac{4}{s}$$

and solving for $Y(s)$ gives:

$$\begin{aligned}Y(s) &= \frac{2s^2 - 3s + 4}{s(s-2)(s-1)} \\ &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \\ &= \frac{2}{s} - \frac{3}{s-1} + \frac{3}{s-2},\end{aligned}$$

where the last two lines represent the decomposition of $Y(s)$ into partial fractions. Taking the inverse Laplace transform gives

$$\boxed{y(t) = 2 - 3e^t + 3e^{2t}.}$$

◀

(b) $y'' + 4y = 6\sin t, \qquad y(0) = 6, \quad y'(0) = 0$

▶ **Solution.** As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation and solving for Y gives:

$$Y(s) = \frac{6s}{s^2+4} + \frac{6}{(s^2+4)(s^2+1)} = \frac{6s}{s^2+4} + \frac{2}{s^2+1} - \frac{2}{s^2+4}.$$

Taking the inverse Laplace transform gives

$$\boxed{y(t) = 6\cos 2t + 2\sin t - \sin 2t.}$$

◀

10. Using the Laplace transform, find the solution of the following differential equations with initial conditions $y(0) = 0$, $y'(0) = 0$:

    (a) $y'' - y = 2\sin t$ $\quad\boxed{y(t) = (1/2)(e^t - e^{-t}) - \sin t}$

    (b) $y'' + 2y' = 5y$ $\quad\boxed{y(t) = 0}$

    (c) $y'' + y = \sin 4t$ $\quad\boxed{y(t) = (1/15)(4\sin t - \sin 4t)}$

    (d) $y'' + y' = 1 + 2t$ $\quad\boxed{y(t) = 1 - e^{-t} + t^2 - t}$

    (e) $y'' + 4y' + 3y = 6$ $\quad\boxed{y(t) = e^{-3t} - 3e^{-t} + 2}$

    (f) $y'' - 2y' = 3(t + e^{2t})$ $\quad\boxed{y(t) = (3/8)(1 - 2t - 2t^2 - e^{2t} + 4te^{2t}}$

    (g) $y'' - 2y' = 20e^{-t}\cos t$ $\quad\boxed{y(t) = 3e^{2t} - 5 + 2e^{-t}(\cos t - 2\sin t)}$

    (h) $y'' + y = 2 + 2\cos t$ $\quad\boxed{y(t) = 2 - 2\cos t + t\sin t}$

    (i) $y'' - y' = 30\cos 3t$ $\quad\boxed{y(t) = 3e^t - 3\cos 3t - \sin 3t}$

11. Solve each of the following homogeneous linear differential equations, using the techniques of Chapter 3 (Characteristic equation).

    (a) $y'' + 3y' + 2y = 0$

    (b) $y'' + 6y' + 13y = 0$

    (c) $y'' + 6y' + 9y = 0$

    (d) $y'' - 2y' - y = 0$

    (e) $8y'' + 4y' + y = 0$

    (f) $2y'' - 7y' + 5y = 0$

    (g) $2y'' + 2y' + y = 0$

    (h) $y'' + .2y' + .01y = 0$

(i) $y'' + 7y' + 12y = 0$

(j) $y'' + 2y' + 2y = 0$

Answers

(a) $y = c_1 e^{-t} + c_2 e^{-2t}$

(b) $y = c_1 e^{-3t}\cos 2t + c_2 e^{-3t}\sin 2t$

(c) $y = (c_1 + c_2 t)e^{-3t}$

(d) $y = c_1 e^{(1+\sqrt{2})t} + c_2 e^{(1-\sqrt{2})t}$

(e) $y = e^{-t/4}(c_1 \cos\frac{t}{4} + c_2 \sin\frac{t}{4})$

(f) $y = c_1 e^{5t/2} + c_2 e^{t}$

(g) $y = e^{-t/2}(c_1 \cos(-t/2) + c_2 \sin(t/2))$

(h) $y = e^{0.1t}(c_1 + tc_2)$

(i) $y = c_1 e^{-4t} + c_2 e^{-3t}$

(j) $y = e^{-t}(c_1 \cos t + c_2 \sin t)$

12. Solve each of the following initial value problems. You may (and should) use the work already done in exercise 10.

(a) $y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3.$

(b) $y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -1.$

(c) $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 5.$

(d) $y'' - 2y' - y = 0, \quad y(0) = 0, \quad y'(0) = \sqrt{2}.$

(e) $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2$

Answers

(a) $y = 2e^{-2t} - e^{-t}$

(b) $y = -\frac{1}{2}e^{-3t}\sin 2t$

(c) $y = (-1 + 2t)e^{-3t}$

(d) $y = \frac{1}{2}(e^{(1+\sqrt{2})t} - e^{(1-\sqrt{2})t})$

(e) $y = 2e^{-t}\sin t$

13. Find a second order linear homogeneous differential equation with constant real coefficients that has the given function as a solution, or explain why there is no such equation.

(a) $e^{-3t} + 2e^{-t}$

(b) $e^{-t}\cos 2t$

(c) $e^{t}t^{-2}$

(d) $5e^{3t/2} + 7e^{-t}$

(e) $2e^{3t}\sin(t/2) - (1/2)e^{3t}\cos(t/2)$

Answers

(a) $y'' + 4y' + 3y = 0$

(b) $y'' + 2y' + 5y = 0$

(c) Not possible since $e^t t^{-2}$ is not an exponential polynomial, that is, it is not a sum of terms of the form $t^k e^a t \cos bt$ or $t^k e^a t \sin bt$ for any choice of $k \geq 0$, $a, b \in \mathbb{R}$. (See Corollary 2, page 185).

(d) $2y'' - y' - 3y = 0$

(e) $4y'' - 24y' + 37y = 0$

14. Use the method of undetermined coefficients (See Section 3.4) to find the general solution of each of the following differential equations.

(a) $y'' - 3y' - 4y = 30e^t$

(b) $y'' - 3y' - 4y = 30e^{4t}$

(c) $y'' - 3y' - 4y = 20\cos t$

(d) $y'' - 2y' + y = t^2 - 1$

(e) $y'' - 2y' + y = 3e^{2t}$

(f) $y'' - 2y' + y = 4\cos t$

(g) $y'' - 2y' + y = 3e^t$

(h) $y'' - 2y' + y = te^t$

Answers

(a) $y = c_1 e^{4t} + c_2 e^{-t} - 5e^t$

(b) $y = c_1 e^{4t} + c_2 e^{-t} + 6te^{4t}$

(c) $y = c_1 e^{4t} + c_2 e^{-t} + (-30/17)\sin t + (-150/51)\cos t$

(d) $y = c_1 e^t + c_2 te^t + t^2 + 4t + 5$

(e) $y = c_1 e^t + c_2 te^t + 3e^{2t}$

(f) $y = c_1 e^t + c_2 te^t + -2\sin t$

(g) $y = c_1 e^t + c_2 te^t + \frac{3}{2}t^2 e^t$

(h) $y = c_e^t + c_2 te^t + \frac{1}{6}t^3 e^t$

15. Solve each of the following initial value problems. You may (and should) use the work already done in exercise 13.

(a) $4t^2 y'' - 7ty' + 6y = 0, \quad y(1) = 1, \quad y'(1) = 2$

(b) $t^2y'' + 5ty' + 4y = 0, \quad y(-1) = 1, \quad y'(-1) = 0$

Answers

(a) $y = \frac{8}{13}t^{-1/2} + \frac{5}{13}t^6$.

(b) $y = t^{-2}(1 - 2\ln|t|)$.