The syllabus for Exam III is Sections 3.6, 4.1–4.3, 4.6, 5.1–5.5, the matrix algebra supplement, and 7.1–7.3. You should review the assigned exercises in these sections. Following is a brief list (not necessarily complete) of terms, skills, and formulas with which you should be familiar.

- Know how to solve the *Cauchy-Euler equation*

$$at^2y'' + bty' + cy = 0$$

where a, b, and c are real constants by means of the roots r_1, r_2 of the indicial equation

$$q(r) = ar(r-1) + br + c = 0.$$

(See Theorem 4, Page 227.)

- If $\{y_1(t),\, y_2(t)\}$ is a fundamental solution set for the homogeneous equation

$$y'' + b(t)y' + c(t)y = 0,$$

know how to use the method of *variation of parameters* to find a particular solution y_p of the non-homogeneous equation

$$y'' + b(t)y' + c(t)y = f(t).$$

In this method, it is not necessary for $f(t)$ to be an exponential polynomial.

Variation of Parameters: Find y_p in the form $y_p = u_1y_1 + u_2y_2$ where u_1 and u_2 are unknown functions whose derivatives satisfy the following two equations:

$$(*) \qquad \begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(t). \end{aligned}$$

Solve the system $(*)$ for u_1' and u_2', and then integrate to find u_1 and u_2.

- Know what it means for a function to have a *jump discontinuity* and to be *piecewise continuous*.

- Know how to piece together solutions on different intervals to produce a solution of one of the initial value problems

$$y' + ay = f(t), \quad y(t_0) = y_0,$$

or

$$y'' + ay' + by = f(t), \quad y(t_0) = y_0,\ y'(t_0) = y_1,$$

where $f(t)$ is a piecewise continuous function on an interval containing t_0.

- Know what the *unit step function* (also called the *Heaviside function*) ($h(t-c)$) and the *on-off switches* ($\chi_{[a,\,b)}$) are:

$$\begin{aligned} h(t-c) = h_c(t) &= \begin{cases} 0 & \text{if } 0 \le t < c, \\ 1 & \text{if } c \ge t \end{cases} \qquad \text{and,} \\ \chi_{[a,\,b)} &= \begin{cases} 1 & \text{if } a \le t < b, \\ 0 & \text{otherwise} \end{cases} = h(t-a) - h(t-b), \end{aligned}$$

and know how to use these two functions to rewrite a piecewise continuous function in a manner which is convenient for computation of Laplace transforms.

- Know the second translation principle (Theorem 8, page 270):

$$\mathcal{L}\left\{f(t-c)h(t-c)\right\} = e^{-cs}F(s)$$

  and how to use it (particulary in the form of Corollary 9, page 271):

$$\mathcal{L}\left\{g(t)h(t-c)\right\} = e^{-cs}\mathcal{L}\left\{g(t+c)\right\}$$

  as a tool for calculating the Laplace transform of piecewise continuous functions.

- Know how to use the second translation principle to compute the inverse Laplace transform of functions of the form $e^{-cs}F(s)$.

- Know how to use the skills in the previous two items in conjunction with the formula for the Laplace transform of a derivative (Theorem 3, Page 278 and Corollary 5, Page 280) to solve differential equations of the form

$$y' + ay = f(t), \quad y(0) = y_0$$

  and

$$y'' + ay' + by = f(t), \quad y(0) = y_0,\ y'(0) = y_1,$$

  where $f(t)$ is a piecewise continuous forcing function.

- Know the formula for the Laplace transform of the Dirac delta function $\delta_c(t)$:

$$\mathcal{L}\left\{\delta_c(t)\right\} = e^{-cs}.$$

  Know how to use this formula to solve differential equations with impulsive forcing function:

$$y' + ay = K\delta_c(t), \quad y(0) = y_0$$

  and

$$y'' + ay' + by = K\delta_c(t), \quad y(0) = y_0,\ y'(0) = y_1.$$

- For a constant coefficient homogeneous linear system

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

  the unique solution is

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0,$$

  where the matrix exponential e^{At} is *defined* by the infinite series

$$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots.$$

  (See Theorem 7.2.3 (page 383) and Corollary 7.3.9 (page 395).) Know how to compute e^{At} from the definition for some simple 2×2 matrices, as is done in the examples on Pages 391–392.

- You should know how to do the algebraic operations on 2×2 matrices. Addition, scalar multiplication, and multiplication of matrices is reviewed in the matrix algebra supplement and in Section 6.1. The following two facts for 2×2 matrices are fundamental. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\det A = ad - bd \neq 0 \iff A \text{ is invertible}$$

and

$$\text{if } \det A \neq 0, \text{ then } A^{-1} = \frac{1}{ad - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then the system of differential equations

$$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

which can also be written as

$$\begin{array}{rcl} y_1' & = & ay_1 + by_2 \\ y_2' & = & cy_1 + dy_2 \end{array} \qquad y_1(0) = c_1, \ y_2(0) = c_2,$$

has the solution

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0).$$

Since

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\},$$

(see Corollary 7, Page 394) the following algorithm can be used to solve $\boldsymbol{y}' = A\boldsymbol{y}$ when the matrix A is a constant matrix.

Algorithm 1. 1. Form the matrix $sI - A = \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}$.

2. Compute the characteristic polynomial

$$p(s) = \det(sI - A) = \det \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix} = s^2 - (a+d)s + (ad - bd).$$

3. Compute

$$(sI - A)^{-1} = \frac{1}{p(s)} \begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} = \begin{bmatrix} \dfrac{s-d}{p(s)} & \dfrac{b}{p(s)} \\ \dfrac{c}{p(s)} & \dfrac{s-a}{p(s)} \end{bmatrix}.$$

4. Compute

$$\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{s-d}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{b}{p(s)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{c}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{s-a}{p(s)}\right\} \end{bmatrix} = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix}.$$

5. The solution $\boldsymbol{y}(t)$ is then

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) = \left(\mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}\right)\boldsymbol{y}(0) = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1h_1(t) + c_2h_2(t) \\ c_1h_3(t) + c_2h_4(t) \end{bmatrix}.$$

The following is a small set of exercises of types identical to those already assigned.

1. Solve each of the following Cauchy-Euler differential equations
   (a) $t^2y'' - 7ty' + 15y = 0$
   (b) $4t^2y'' - 8ty' + 9y = 0$
   (c) $t^2y'' - 12y = 0$
   (d) $t^2y'' + ty' + 16y = 0$
   (e) $4t^2y'' - 7ty' + 6y = 0$
   (f) $t^2y'' + 5ty' + 4y = 0$

2. Find a solution of each of the following differential equations by using the method of variation of parameters. In each case, $\mathcal{S}$ denotes a fundamental set of solutions of the associated homogeneous equation.
   (a) $y'' + 2y' + y = t^{-1}e^{-t}; \quad \mathcal{S} = \left\{e^{-t},\, te^{-t}\right\}$
   (b) $y'' + 9y = 9\sec 3t; \quad \mathcal{S} = \{\cos 3t,\, \sin 3t\}$
   (c) $t^2y'' + 2ty' - 6y = t^2; \quad \mathcal{S} = \left\{t^2,\, t^{-3}\right\}$
   (d) $y'' + 4y' + 4y = t^{1/2}e^{-2t}; \quad \mathcal{S} = \left\{e^{-2t},\, te^{-2t}\right\}$

3. Graph each of the following piecewise continuous functions.
   (a) $f(t) = \begin{cases} 2 & \text{if } 0 \le t < 1, \\ 3 - 2t & \text{if } t \ge 1. \end{cases}$
   (b) $f(t) = e^{-2(t-1)}h(t-1)$
   (c) $f(t) = t\chi_{[0,2)} - 3\chi[2,4) + (t-4)^2h(t-4)$

4. Compute the Laplace transform of each of the following functions.
   (a) $f(t) = t^2h(t-2)$
   (b) $f(t) = (t-1)^2\chi_{[2,\,4)}$
   (c) $f(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ t^2 + t - 1 & \text{if } 1 \le t < 2 \\ -2 & \text{if } t \ge 2. \end{cases}$

5. Compute the inverse Laplace transform of each of the following functions.
   (a) $F(s) = \dfrac{1 - e^{-s}}{s}$
   (b) $F(s) = \dfrac{1 + e^{-\pi s}}{s^2 + 1}$
   (c) $F(s) = \dfrac{se^{-2s}}{s^2 - 9}$
   (d) $F(s) = (1 + e^{-\pi s/2})\dfrac{s - 1}{s^2 + 2s + 10}$

6. Solve each of the following differential equations.

   (a) $y' + 5y = \begin{cases} 2 & \text{if } 0 \le t < 2 \\ t+1 & \text{if } t \ge 2 \end{cases} \qquad y(0) = 0.$

   (b) $y'' + 4y = (t+1)h(t-2), \quad y(0) = 0,\ y'(0) = 1.$

   (c) $y'' + 4y = 4\delta_2(t), \quad y(0) = -1,\ y'(0) = 1.$

7. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

   (a) Compute $(sI - A)$ and $(sI - A)^{-1}$.

   (b) Find $\mathcal{L}^{-1}((sI - A)^{-1})$.

   (c) What is e^{At}?

   (d) Solve the system $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

8. Solve the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$ where $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

9. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

10. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -3 \\ 3 & 6 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Answers

1. (a) $y = c_1 t^3 + c_2 t^5$
   (b) $y = t^{3/2}(c_1 + c_2 \ln|t|)$
   (c) $y = c_1 t^4 + c_2 t^{-3}$
   (d) $y = c_1 \cos(4\ln|t|) + c_2 \sin(4\ln|t|)$
   (e) $y = c_1 t^{-1/2} + c_2 t^6$
   (f) $y = t^{-2}(c_1 + c_2 \ln|t|)$
2. (a) $y_p(t) = (t\ln t)e^{-t}$
   (b) $y_p(t) = \frac{1}{2}(\ln|\cos 3t|)\cos 3t + \frac{3}{2} t \sin 3t$
   (c) $y_p(t) = (1/5)t^2 \ln|t|$
   (d) $y_p(t) = (4/15)t^{5/2}e^{-2t}$
4. (a) $F(s) = e^{-s}\left(\dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right)$
   (b) $F(s) = e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right) - e^{-4s}\left(\dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}\right)$
   (c) $F(s) = \dfrac{1}{s^2} + e^{-s}\left(\dfrac{2}{s^3} + \dfrac{2}{s^2}\right) - e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{5}{s^2} + \dfrac{7}{s}\right)$
5. (a) $f(t) = 1 - h(t-1)$
   (b) $f(t) = (1 - h(t-\pi))\sin t$
   (c) $f(t) = \frac{1}{2}\left(e^{3(t-1)} + e^{-3(t-1)}\right)h(t-1)$
   (d) $f(t) = e^{-t}\left(\cos 3t - \frac{2}{3}\sin 3t\right) + e^{-(t-\pi/2)}\left(\cos 3(t-\pi/2) - \frac{2}{3}\sin 3(t-\pi/2)\right)h(t-\pi/2)$
6. (a) $y(t) = \frac{2}{5}(1 - e^{-5t}) + \left(e^{-5(t-2)} + 5(t-2) - 6\right)h(t-2)$
   (b) $y(t) = \frac{1}{2}\sin 2t + h(t-2)\left((t-2) + 3 - 3\cos 2(t-2) - \frac{1}{2}\sin 2(t-2)\right)$
   (c) $y(t) = \frac{1}{2}\sin 2t - \cos 2t + 2h(t-2)\sin 2(t-2)$
7. (a) $sI - A = \begin{bmatrix} s-1 & 2 \\ 3 & s-2 \end{bmatrix}$; $(sI-A)^{-1} = \begin{bmatrix} \dfrac{s-2}{(s-4)(s+1)} & \dfrac{-2}{(s-4)(s+1)} \\ \dfrac{-3}{(s-4)(s+1)} & \dfrac{s-1}{(s-4)(s+1)} \end{bmatrix}$
   (b) $\frac{1}{5}\begin{bmatrix} 2e^{4t} + 3e^{-t} & -2e^{4t} + 2e^{-t} \\ -3e^{4t} + 3e^{-t} & -3e^{4t} + 8e^{-t} \end{bmatrix}$ (c) e^{At} is same as $\mathcal{L}^{-1}((sI-A)^{-1})$.
   (d) $\boldsymbol{y}(t) = \frac{1}{5}\begin{bmatrix} -8e^{4t} + 3e^{-t} \\ 21e^{-t} - 6e^{4t} \end{bmatrix}$
8. $\boldsymbol{y}(t) = \frac{1}{6}\begin{bmatrix} (5c_1 - c_2)e^{4t} + (c_1 + c_1)e^{-2t} \\ (-5c_1 + c_2)e^{4t} + (5c_2 + 5c_1)e^{-2t} \end{bmatrix}$
9. $\boldsymbol{y}(t) = \frac{1}{2}\begin{bmatrix} 1 + e^{4t} \\ -2 + 2e^{4t} \end{bmatrix}$
10. $\boldsymbol{y}(t) = \begin{bmatrix} e^{3t} + 3te^{3t} \\ -2e^{3t} - 3te^{3t} \end{bmatrix}$