Math 2065 Section 1
Final Exam Review Sheet

The final exam will be on Wednesday, May 12 from 10:00 AM to 12:00 Noon in the normal classroom. The exam is closed book, but you will be provided with the usual table of Laplace transforms.

The final exam is comprehensive, and thus any of the material we covered is a valid source for questions.

You should collect each of the review sheets and exams as sources for study for the final exam. Each of these is posted on the class web site, in case you have misplaced them. A good strategy for study is to do the review sheets and exams *without* looking at the answers. If you then compare with the answer sheets, you can identify the areas in which you need additional work. Some additional review exercises are included here, of exactly the same type as found in your text and on the previous review sheets.

Review Exercises

Solve each of the following differential equations.

1. $y' = t - 2y$
2. $y' = t - 4ty$
3. $y' + \dfrac{4}{t}y = t^4$
4. $yy' = (t-1)^2$
5. $y' = 1 + t + y^2 + ty^2$
6. For the equation $y'(1-t) = y$,
   (a) Find the general solution.
   (b) Find the particular solution with $y(2) = 1$, and give its interval of existence.
7. Consider the initial value problem $ty' = e^t - y$, $y(1) = e$.
   (a) Without solving the equation, give the domain of existence of the solution, as guaranteed by the existence and uniqueness theorem.
   (b) Now solve the equation and see if your answer is indeed defined on the interval you found in Part (a).
8. $y'' - 3y' + 2y = 0$
9. $y'' + 2y' + 2y = 0$

10. $y'' + 4y' + 4y = 0$

11. $y'' + 6y' + 9y = 0$

12. $y'' - 6y' + 13y = 0$

13. $y'' + 16y = 0$

14. $y'' - 2y' - 3y = 0, \quad y(0) = 0, \quad y'(0) = 1$

15. $y'' + 6y' + 13y = 0, \quad y(0) = 1, \quad y'(0) = -1$

16. $y'' - 2y' - y = 0$

17. $y'' + 2y' - 15y = 0$

18. $t^2y'' + 2ty' - 6y = 0$

19. $3t^2y'' + 11ty' - 3y = 0$

20. $t^2y'' + 9ty' + 17y = 0$

21. $t^2y'' - 3ty' + 4y = 0$

22. $y'' - 2y' + y = 3e^{2t}$

23. $y'' + 2y' + y = 2e^{-t}$

24. $y'' - y' - 2y = -9e^{-t}$

25. $y'' - 2y' + y = \frac{e^t}{t^5}$

26. $y'' + \frac{1}{t}y' - \frac{1}{t^2}y = \ln t$, $(t > 0)$. You may assume that a fundamental set for the associated homogeneous equation is $\{\varphi_1(t) = t, \varphi_2(t) = t^{-1}\}$.

Find the Laplace transform of each of the following functions.

27. t^2e^{-9t}

28. $e^{2t} - t^3 + t^2 - \sin 5t$

29. $t \cos 6t$

30. $2 \sin t + 3 \cos 2t$

31. $e^{-5t} \sin 6t$

32. $t^2 \cos at$ where a is a constant

33. $f(t) = \begin{cases} 1 & \text{if } 0 \le 0 < 2, \\ -1 & \text{if } 2 \le t < 4, \text{ and} \\ 0 & \text{if } t \ge 4. \end{cases}$

34. $f(t) = (t^2 - 100)h(t - 10)$

Find the inverse Laplace transform of each of the following functions.

35. $\dfrac{1}{s^2 - 10s + 9}$

36. $\dfrac{2s - 18}{s^2 + 9}$

37. $\dfrac{2s + 18}{s^2 + 25}$

38. $\dfrac{s + 3}{s^2 + 5}$

39. $\dfrac{s - 3}{s^2 - 6s + 25}$

40. $\dfrac{1}{s(s^2 + 4)}$

41. $\dfrac{1}{s^2(s + 1)^2}$

42. $\dfrac{1 - e^{-s}}{s}$

43. $\dfrac{1 + e^{-\pi s}}{s^2 + 1}$

44. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

   (a) Compute $(sI - A)$ and $(sI - A)^{-1}$.

   (b) Find $\mathcal{L}^{-1}((sI - A)^{-1})$.

   (c) What is e^{At}?

   (d) Solve the system $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

45. Solve the matrix differential equation $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

46. Solve the initial value problem:

$$\mathbf{y}' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

47. Solve the initial value problem:

$$\mathbf{y}' = \begin{bmatrix} 0 & -3 \\ 3 & 6 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

48. Consider a pond with 1000 cubic meters of water. There is a stream flowing out from the pond at a rate of 10 cubic meters a day. Nearby is a field which is regularly irrigated and fertilized. Each day, 10 cubic meters of water from the field enters the pond, and this is contaminated with 3 kilograms of ammonium nitrate per cubic meter. Write down a differential equation for the amount of ammonium nitrate in the pond at time t. Assume the ammonium nitrate is perfectly mixed and ignore the effect of rain and evaporation. Do not solve the equation.

Answers

1. $y = \frac{1}{2}t - \frac{1}{4} + ce^{-2t}$
2. $4y = 1 + ce^{-2t^2}$
3. $y = \frac{c}{t^4} + \frac{1}{9}t^5$
4. $3y^2 - 2(t-1)^3 = c$
5. $\arctan y - t - \frac{t^2}{2} = c$
6. (a) $y = \frac{c}{1-t}$

   (b) $y = \frac{1}{t-1}$. The interval of existence is $(1, \infty)$
7. (a) $(0, \infty)$ (b) $y(t) = \frac{e^t}{t}$
8. $y = c_1e^t + c_2e^{2t}$
9. $y = e^{-t}(c_1 \cos t + c_2 \sin t)$
10. $y = c_1e^{-2t} + c_2te^{-2t}$
11. $y = c_1e^{-3t} + c_2te^{-3t}$
12. $y = c_1e^{3t} \cos 2t + c_2e^{3t} \sin 2t$
13. $y = c_1 \cos 4t + c_2 \sin 4t$
14. $y = \frac{1}{4}(e^{3t} - e^{-t})$
15. $y = e^{-3t}(\cos 2t + \sin 2t)$
16. $y = c_1e^{(1+\sqrt{2})t} + c_2e^{(1-\sqrt{2})t}$
17. $y = c_1e^{3t} + c_2e^{-5t}$
18. $y = c_1t^{-3} + c_2t^2$
19. $y = c_1t^{1/3} + c_2t^{-3}$
20. $y = t^{-4}(c_1 \cos(\ln |t|) + c_2 \sin(\ln |t|))$
21. $y = c_2t^2 + c_2t^2 \ln |t|$
22. $y = c_1e^t + c_2te^t + 3e^{2t}$
23. $y = c_1e^{-t} + c_2te^{-t} + t^2e^{-t}$
24. $y = c_1e^{-t} + c_2e^{2t} + 3te^{-t}$
25. $y = c_1e^t + c_2te^t + \frac{1}{12}t^{-3}e^t$
26. $y = c_1t + c_2t^{-1} + \frac{t^2}{3} \ln t - \frac{4}{9}t^2$
27. $\frac{2}{(s+9)^3}$
28. $\frac{1}{s-2} - \frac{6}{s^4} + \frac{2}{s^3} - \frac{5}{s^2+25}$
29. $\frac{s^2-36}{(s^2+36)^2}$
30. $\frac{2}{s^2+1} + \frac{3s}{s^2+4}$
31. $\frac{6}{(s+5)^2+36}$
32. $\frac{2s^3-6sa^2}{(s^2+a^2)^2}$
33. $\frac{1-2e^{-2s}+e^{-4s}}{s}$
34. $\frac{2e^{-10s}}{s^3} + \frac{20e^{-10s}}{s^2}$

35. $\frac{1}{8}(e^{9t} - e^{t})$

36. $2\cos 3t - 6\sin 3t$

37. $2\cos 5t + \frac{18}{5}\sin 5t$

38. $\cos\sqrt{5}t + \frac{3}{\sqrt{5}}\sin\sqrt{5}t$

39. $e^{3t}\cos 4t$

40. $\frac{1}{4}(1 - \cos 2t)$

41. $te^{-t} + 2e^{-t} + t - 2$

42. $1 - h(t-1)$

43. $\sin t\,(1 - h(t-\pi))$

44. (a) $sI - A = \begin{bmatrix} s-1 & 2 \\ 3 & s-2 \end{bmatrix}$; $(sI-A)^{-1} = \begin{bmatrix} \frac{s-2}{(s-4)(s+1)} & \frac{-2}{(s-4)(s+1)} \\ \frac{-3}{(s-4)(s+1)} & \frac{s-1}{(s-4)(s+1)} \end{bmatrix}$

(b) $\frac{1}{5}\begin{bmatrix} 2e^{4t} + 3e^{-t} & -2e^{4t} + 2e^{-t} \\ -3e^{4t} + 3e^{-t} & -3e^{4t} + 8e^{-t} \end{bmatrix}$ (c) e^{At} is same as $\mathcal{L}^{-1}((sI-A)^{-1})$.

(d) $\mathbf{y}(t) = \frac{1}{5}\begin{bmatrix} -8e^{4t} + 3e^{-t} \\ 21e^{-t} - 6e^{4t} \end{bmatrix}$

45. $\mathbf{y}(t) = \frac{1}{6}\begin{bmatrix} (5c_1 - c_2)e^{4t} + (c_1 + c_1)e^{-2t} \\ (-5c_1 + c_2)e^{4t} + (5c_2 + 5c_1)e^{-2t} \end{bmatrix}$

46. $\mathbf{y}(t) = \frac{1}{2}\begin{bmatrix} 1 + e^{4t} \\ -2 + 2e^{4t} \end{bmatrix}$

47. $\mathbf{y}(t) = \begin{bmatrix} e^{3t} + 3te^{3t} \\ -2e^{3t} - 3te^{3t} \end{bmatrix}$

48. If $y(t)$ denotes the number of kilograms of ammonium nitrate at time t, then $y'(t) = 30 - \frac{y(t)}{100}$