Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the main partial fraction decomposition theorem have been appended to the exam.

In Exercises 1 – 8, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $y' + 3y = 3e^{2t} - 2e^{-3t}, \quad y(0) = 2.$
2. [**12 Points**] $y' = -t/y, \quad y(0) = 4.$
3. [**10 Points**] $y'' + 10y' + 29y = 0.$
4. [**10 Points**] $4y'' + 12y' + 9y = 0.$
5. [**12 Points**] $y'' - 4y' + 3y = 0 = 0, \quad y(0) = 1,\ y'(0) = -1.$
6. [**12 Points**] $y'' + 2y' - 8y = 4e^{2t}.$
7. [**10 Points**] $2t^2y'' + 5ty' - 2y = 0.$
8. [**12 Points**] $y'' + 25y = 2\delta(t - \pi), \quad y(0) = 2,\ y'(0) = 3.$ (Recall that $\delta(t)$ is the Dirac delta function.)
9. [**12 Points**] Find a particular solution of the differential equation

$$y'' + \frac{2}{t}y' - \frac{6}{t^2}y = 80t^3,$$

given the fact that the general solution of the associated homogeneous equation is

$$y_h = c_1t^2 + c_2t^{-3}.$$

10. [**8 Points**] Find the Laplace transform $F(s)$ of the following function $f(t)$.

$$f(t) = \begin{cases} 2 & \text{if } 0 \le t < 3, \\ t^2 - 9 & \text{if } t \ge 3. \end{cases}$$

11. [**14 Points**] Compute each of the following inverse Laplace transforms.

(a) $\mathcal{L}^{-1}\left\{\dfrac{1 - 2s}{(s-3)^2 + 5}\right\}$

(b) $\mathcal{L}^{-1}\left\{\dfrac{4s}{(s+1)(s^2 - 4)}\right\}$

12. [**18 Points**] Let $A = \begin{bmatrix} 2 & 4 \\ -4 & -6 \end{bmatrix}$.

    (a) Compute $(sI - A)^{-1}$.

    (b) Find $e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\}$.

    (c) Find the general solution of the system $\mathbf{y}' = A\mathbf{y}$.

    (d) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

13. [**12 Points**] A tank initially contains 2000 gallons of water with 100 pounds of salt dissolved. Brine (a water-salt mixture) containing 0.8 pounds of salt per gallon flows into the tank at the rate of 5 gal/min, and the mixture (which is assumed to be perfectly mixed) flows out of the tank at the same rate of 5 gal/min.

    (a) Find the amount of salt $y(t)$ in the tank at time t.

    (b) How much salt does the tank contain after 3 hours?

    (c) What is $\lim_{t\to\infty} y(t)$?

Laplace Transform Table

	$f(t)$		$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. Then there is a unique number A_1 and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 and the polynomial $p_1(s)$ are given by

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d is an irreducible quadratic that is factored completely out of $q(s)$. Then there is a unique linear term $B_1 s + C_1$ and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ is a complex root of s^2+cs+d then $B_1 s + C_1$ and the polynomial $p_1(s)$ are given by

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$