

$$11. \quad t^2 y'' + ty' + 4y = 0, \quad y(1) = -3, \quad y'(1) = 4$$

$$12. \quad t^2 y'' - 4ty' + 6y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

3.5 Undetermined Coefficients

In this section and the next we consider the nonhomogeneous differential equation

$$\mathbf{L}y = f, \tag{1}$$

where f is a nonzero function. The general theory developed in Sections 3.1 and 3.2, specifically Theorems 3.1.4 and 3.2.4, gives the strategy for solving Equation (1): First, we find the solution set, $\mathcal{S}_{\mathbf{L}}^0$, to the associated homogeneous equation $\mathbf{L}y = 0$. Second, we find a particular solution φ_p to Equation (1). Then the general solution takes the form

$$\varphi_p + \varphi_h,$$

where $\varphi_h \in \mathcal{S}_{\mathbf{L}}^0$. Previous sections have addressed the question of finding the solution set to a linear second order differential equation in some special circumstances. Thus our efforts now turn to finding a particular solution to Equation (1).

In this section we will describe a method, known as the **method of undetermined coefficients**, for finding a particular solution to

$$ay'' + by' + cy = f(t) \tag{2}$$

in the case where $f \in \mathcal{E}$ is an elementary function and a , b , and c are real numbers with $a \neq 0$. The general case will be considered in the next section.

Recall from Chapter 2 that an elementary function is a sum of constant multiples of the following types of simple elementary functions:

$$t^k e^{\alpha t}, \quad t^k e^{\alpha t} \cos \beta t, \quad t^k e^{\alpha t} \sin \beta t,$$

where α and β are real numbers and $k = 0, 1, 2, \dots$. As we shall see any solution to Equation (2) is again an elementary function and therefore any particular solution φ_p can be expressed in the following way

$$\varphi_p = C_1 \varphi_1 + \dots + C_n \varphi_n, \tag{3}$$

where each φ_i , $i = 1, \dots, n$, is a simple elementary function. If each of the φ_i 's is not a solution to the associated homogeneous equation $ay'' + by' + cy = 0$ we will call Equation (3) the **form of the particular solution**. The method of undetermined coefficients allows one to determine the simple elementary functions that appear as terms in Equation (3). The coefficients of these term are then determined by substitution into Equation (2).

Let's consider the necessary details. Let $y(t)$ be the unique solution of Equation (2) subject to the initial conditions $y(0) = y_0$ and $y'(0) = y_1$, and, as usual, we let $Y(s) = \mathcal{L}(y(t))$. Applying the Laplace transform to both sides of Equation (2) gives

$$a(s^2Y - sy_1 - y_1) + b(sY - y_0) + cY = F(s) = \frac{R(s)}{Q(s)},$$

and solving for Y we obtain

$$Y = \frac{y_0(as + b) + ay_1}{p(s)} + \frac{R(s)}{p(s)Q(s)}, \quad (4)$$

which we write as $Y_1(s) + Y_2(s)$, where $Y_1(s)$ is the first term and $Y_2(s)$ is the second term. Since Y is a proper rational function y is an elementary function. The first term in Equation (4), $Y_1(s)$, has inverse Laplace transform that is part of the solution to the associated homogeneous equation $ay'' + by' + cy = 0$. Since our focus is on finding the form of a particular solution we can ignore this contribution and concentrate on the second term $Y_2(s)$. The form of the partial fraction decomposition (see Page 96) of $Y_2(s)$ is completely determined by the factorization of the denominator $p(s)Q(s)$. The inverse Laplace transform of Y_2 is thus a sum of simple elementary functions. Some of these simple elementary functions may be included in φ_h and therefore ignored. It is the remaining terms that lead to the form of the particular solution. The way to proceed should become clear once we have illustrated the method with some simple examples.

Example 3.5.1. Find a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3e^{-t}$.

► **Solution.** In this example, $f(t) = 3e^{-t}$ so that $F(s) = \frac{3}{s+1} = \frac{R(s)}{Q(s)}$ and $p(s) = s^2 + 4s - 5 = (s+5)(s-1)$; hence $p(s)Q(s) = (s+5)(s-1)(s+1)$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{s+1},$$

and hence

$$y(t) = Ae^{-5t} + Be^t + Ce^{-t}.$$

The first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = Ce^{-t}$ where C is a constant, which can be determined by substitution into the original equation:

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p'(t) - 5\varphi_p(t) &= Ce^{-t} - 4Ce^{-t} - 5Ce^{-t} \\ &= -8Ce^{-t} \\ &= 3e^{-t}.\end{aligned}$$

Thus, we must have $-8C = 3$ so that $C = -3/8$ and $\varphi_p(t) = (-3/8)e^{-t}$. The general solution to $y'' + 4y' - 5y = 3e^{-t}$ is then

$$y(t) = Ae^{-5t} + Be^t - \frac{3}{8}e^{-t},$$

where A and B are arbitrary real constants. ◀

Example 3.5.2. Find a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3te^{-t}$.

► **Solution.** The only difference between this and the previous example is that now $f(t) = 3te^{-t}$ so that $F(s) = \frac{3}{(s+1)^2} = \frac{R(s)}{Q(s)}$. Hence $p(s)Q(s) = (s+5)(s-1)(s+1)^2$ which gives a partial fraction expansion for $Y(s)$ of the form

$$Y(s) = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2},$$

which then implies that

$$y(t) = Ae^{-5t} + Be^t + C_1e^{-t} + C_2te^{-t}.$$

As above, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1e^{-t} + C_2te^{-t}$ where C_1 and C_2 are constants, which can be determined by substitution into the original equation, as follows. First compute the derivatives of $\varphi_p(t)$:

$$\begin{aligned}\varphi_p(t) &= C_1e^{-t} + C_2te^{-t} \\ \varphi_p'(t) &= (-C_1 + C_2)e^{-t} - C_2te^{-t} \\ \varphi_p''(t) &= (C_1 - 2C_2)e^{-t} + C_2te^{-t}.\end{aligned}$$

Now substitute these into the original equation:

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p'(t) - 5\varphi_p(t) &= (-8C_1 + 2C_2)e^{-t} - 8C_2te^{-t} \\ &= 3te^{-t}.\end{aligned}$$

Setting $t = 0$ gives the equation $-8C_1 + 2C_2 = 0$, while comparing the coefficients of te^{-t} gives an equation $-8C_2 = 3$. Hence the coefficients C_1 and C_2 satisfy the system of equations

$$\begin{aligned} -8C_1 + 2C_2 &= 0 \\ -8C_2 &= 3. \end{aligned}$$

Therefore, $C_2 = -3/8$ and $C_1 = C_2/4 = -3/32$, and we conclude that a particular solution $\varphi_p(t)$ to $y'' + 4y' - 5y = 3te^{-t}$ is given by

$$\varphi_p(t) = -\frac{3}{32}e^{-t} - \frac{3}{8}te^{-t}$$

and then general solution is

$$y(t) = \varphi_h(t) + \varphi_p(t) = Ae^{-5t} + Be^t - \frac{3}{32}e^{-t} - \frac{3}{8}te^{-t},$$

where A and B are arbitrary constants. ◀

Example 3.5.3. Find a particular solution $\varphi_p(t)$ to $y'' - 4y' - 5y = 3e^{-t}$.

► **Solution.** In this example, as in the first example, $f(t) = 3e^{-t}$ so that $F(s) = \frac{3}{s+1} = \frac{R(s)}{Q(s)}$. But $p(s) = s^2 - 4s - 5 = (s-5)(s+1)$; hence $p(s)Q(s) = (s-5)(s+1)^2$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A}{s-5} + \frac{B}{s+1} + \frac{C}{(s+1)^2},$$

and hence

$$y(t) = Ae^{5t} + Be^{-t} + Cte^{-t}.$$

The first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = Cte^{-t}$ where C is a constant, which can be determined by substitution into the original equation:

$$\begin{aligned} \varphi_p''(t) - 4\varphi_p'(t) - 5\varphi_p(t) &= C(-2+t)e^{-t} - 4C(1-t)e^{-t} - 5Cte^{-t} \\ &= -6Ce^{-t} \\ &= 3e^{-t}. \end{aligned}$$

Thus, we must have $-6C = 3$ so that $C = -1/2$ and $\varphi_p(t) = (-1/2)te^{-t}$. The general solution to $y'' - 4y' - 5y = 3e^{-t}$ is then

$$y(t) = Ae^{-5t} + Be^t - \frac{1}{2}te^{-t},$$

where A and B are arbitrary real constants. ◀

Example 3.5.4. Find a particular solution $\varphi_p(t)$ to $y'' + 2y' + y = 3e^{-t}$.

► **Solution.** Also in this example, $f(t) = 3e^{-t}$ so that $F(s) = \frac{3}{s+1} = \frac{R(s)}{Q(s)}$. But $p(s) = s^2 + 2s + 1 = (s+1)^2$; hence $p(s)Q(s) = (s+1)^3$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1}{s+1} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)^3},$$

and hence

$$y(t) = A_1 e^{-t} + A_2 t e^{-t} + (A_3/2) t^2 e^{-t}.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C t^2 e^{-t}$ where C is a constant, which, as earlier, can be determined by substitution into the original equation:

$$\begin{aligned} \varphi_p''(t) + 2\varphi_p'(t) + \varphi_p(t) &= C(2 - 4t + t^2)e^{-t} + 2C(2t - t^2)e^{-t} + C t^2 e^{-t} \\ &= 2C e^{-t} \\ &= 3e^{-t}. \end{aligned}$$

Thus, $C = 3/2$ and $\varphi_p(t) = (3/2)t^2 e^{-t}$. The general solution to $y'' + 2y' + y = 3e^{-t}$ is then

$$y(t) = A_1 e^{-t} + A_2 t e^{-t} + \frac{3}{2} t^2 e^{-t},$$

where A_1 and A_2 are arbitrary real constants. ◀

Example 3.5.5. Find a particular solution $\varphi_p(t)$ to $y'' + 2y' + 5y = 3 \sin 2t$.

► **Solution.** In this example, $f(t) = 3 \sin 2t$ so that $F(s) = \frac{6}{s^2 + 4} = \frac{R(s)}{Q(s)}$, while $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$; hence $p(s)Q(s) = ((s+1)^2 + 4)(s^2 + 4)$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1 s + B_1}{(s+1)^2 + 4} + \frac{A_2 s + B_2}{s^2 + 4},$$

and hence (using Formulas 6) and 7) of Table C.2)

$$y(t) = \tilde{A}_1 e^{-t} \cos 2t + \tilde{B}_1 e^{-t} \sin 2t + A_2 \cos 2t + (B_2/2) \sin 2t.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1 \cos 2t + C_2 \sin 2t$ where C_1 and C_2 are constants to be determined by substitution into the original equation:

$$\begin{aligned}\varphi_p''(t) + 2\varphi_p'(t) + 5\varphi_p(t) &= (-4C_1 + 4C_2 + 5C_1) \cos 2t + (-4C_2 - 4C_1 + 5C_2) \sin 2t \\ &= (C_1 + 4C_2) \cos 2t + (-4C_1 + C_2) \sin 2t \\ &= 3 \sin 2t.\end{aligned}$$

Thus, C_1 and C_2 satisfy the system of linear equations:

$$\begin{aligned}C_1 + 4C_2 &= 0 \\ -4C_1 + C_2 &= 3.\end{aligned}$$

Solving these equations for C_1 and C_2 gives

$$C_1 = -\frac{12}{17}, \quad C_2 = \frac{3}{17},$$

which implies that a particular solution $\varphi_p(t)$ to $y'' + 2y' + 5y = 3 \sin 2t$ is

$$\varphi_p(t) = -\frac{12}{17} \cos 2t + \frac{3}{17} \sin 2t,$$

and the general solution is then

$$y(t) = A_1 e^{-t} \cos 2t + A_2 e^{-t} \sin 2t - \frac{12}{17} \cos 2t + \frac{3}{17} \sin 2t,$$

where A_1 and A_2 are arbitrary real constants. ◀

Example 3.5.6. Find a particular solution $\varphi_p(t)$ to $y'' + 4y = 3 \sin 2t$.

► **Solution.** As in the previous example, $f(t) = 3 \sin 2t$ so that $F(s) = \frac{6}{s^2 + 4} = \frac{R(s)}{Q(s)}$, but now $p(s) = s^2 + 4$; hence $p(s)Q(s) = (s^2 + 4)^2$. Since this is the denominator of $Y(s)$ we conclude that

$$Y(s) = \frac{A_1 s + B_1}{s^2 + 4} + \frac{A_2 s + B_2}{(s^2 + 4)^2},$$

and hence (using Formulas from the Table of Convolutions)

$$y(t) = \tilde{A}_1 \cos 2t + \tilde{B}_1 \sin 2t + C_1 t \cos 2t + C_2 t \sin 2t.$$

As in the previous examples, the first two terms are included in $\varphi_h(t)$ so we conclude that $\varphi_p(t) = C_1 t \cos 2t + C_2 t \sin 2t$ where C_1 and C_2 are constants to be determined by

Table 3.1: Form of a particular solution $\varphi_p(t)$

$p(s)$	$f(t)$	$Q(s)$	form of $\varphi_p(t)$
$(s+5)(s-1)$	$3e^{-t}$	$s+1$	Ce^{-t}
$(s+5)(s-1)$	$3te^{-t}$	$(s+1)^2$	$(C_1 + C_2t)e^{-t}$
$(s-5)(s+1)$	$3e^{-t}$	$s+1$	Cte^{-t}
$(s+1)^2$	$3e^{-t}$	$s+1$	Ct^2e^{-t}
$(s+1)^2 + 4$	$3\sin 2t$	$s^2 + 4$	$C_1 \cos 2t + C_2 \sin 2t$
$s^2 + 4$	$3\sin 2t$	$s^2 + 4$	$C_1t \cos 2t + C_2t \sin 2t$

substitution into the original equation (the details of the substitution are left to the reader):

$$\begin{aligned}\varphi_p''(t) + 4\varphi_p(t) &= 4C_2 \cos 2t - 4C_1 \sin 2t \\ &= 3 \sin 2t.\end{aligned}$$

Solving for C_1 and C_2 gives

$$C_1 = -\frac{3}{4}, \quad C_2 = 0,$$

which implies that a particular solution $\varphi_p(t)$ to $y'' + 4y = 3 \sin 2t$ is

$$\varphi_p(t) = -\frac{3}{4}t \cos 2t,$$

and the general solution is then

$$y(t) = A_1 \cos 2t + A_2 \sin 2t - \frac{3}{4}t \cos 2t,$$

where A_1 and A_2 are arbitrary real constants. ◀

We now summarize the calculations of the previous examples in Table 3.1. In this table, we are tabulating the *form* of a particular solution $\varphi_p(t)$ of the differential equation

$$ay'' + by' + cy = f(t)$$

as it relates to the characteristic polynomial $p(s) = as^2 + bs + c$, the forcing function $f(t)$, and the denominator $Q(s)$ of the Laplace transform $F(s) = R(s)/Q(s)$ of $f(t)$.

Notice that so long as $p(s)$ and $Q(s)$ do not have a common root (as in rows 1, 2, and 5 of Table 3.1), then the form of $\varphi_p(t)$ is exactly similar to that of $f(t)$, while if $p(s)$ and $Q(s)$ have a common root (either real or complex), then the form of $\varphi_p(t)$ is adjusted by multiplying by either t (if the common root is a simple root of $p(s)$, as in rows 3 and 6 of the table) or t^2 (if the common root is a double root of $p(s)$, as in row 4 of the table). These observations are formalized in the following theorem.

Theorem 3.5.7 (Undetermined Coefficients). Let $p(s) = as^2 + bs + c$ be the characteristic polynomial of the nonhomogeneous constant coefficient differential equation

$$ay'' + by' + cy = f(t).$$

The form of a particular solution $\varphi_p(t)$ of this equation is determined by $f(t)$ and $p(s)$ in the following cases.

1. $f(t) = (A_0 + A_1t + \cdots + A_k t^k)e^{\alpha t}$.

(a) If $p(\alpha) \neq 0$, then the form of $\varphi_p(t)$ is

$$\varphi_p(t) = (C_0 + C_1t + \cdots + C_k t^k)e^{\alpha t}.$$

(b) If $p(s) = a(s - \alpha)(s - r)$ with $r \neq \alpha$, i.e., α is a simple root of $p(s)$, then the form of $\varphi_p(t)$ is

$$\varphi_p(t) = t(C_0 + C_1t + \cdots + C_k t^k)e^{\alpha t}.$$

(c) If $p(s) = a(s - \alpha)^2$, i.e., α is a double root of $p(s)$, then the form of $\varphi_p(t)$ is

$$\varphi_p(t) = t^2(C_0 + C_1t + \cdots + C_k t^k)e^{\alpha t}.$$

2. $f(t) = (A_0 + A_1t + \cdots + A_k t^k)e^{\alpha t} \cos \beta t + (A'_0 + A'_1t + \cdots + A'_k t^k)e^{\alpha t} \sin \beta t$.

(a) If $p(\alpha + i\beta) \neq 0$ then the form of $\varphi_p(t)$ is

$$\varphi_p(t) = (C_0 + C_1t + \cdots + C_k t^k)e^{\alpha t} \cos \beta t + (C'_0 + C'_1t + \cdots + C'_k t^k)e^{\alpha t} \sin \beta t.$$

(b) If $p(\alpha + i\beta) = 0$ then the form of $\varphi_p(t)$ is

$$\varphi_p(t) = t(C_0 + C_1t + \cdots + C_k t^k)e^{\alpha t} \cos \beta t + t(C'_0 + C'_1t + \cdots + C'_k t^k)e^{\alpha t} \sin \beta t.$$

The form of $\varphi_p(t)$ means that $C_1, C_2, \dots, C_k$, and $C'_1, C'_2, \dots, C'_k$ are initially undetermined coefficients which are computed by substitution into the differential equation.

Example 3.5.8. Determine the form of a particular solution $\varphi_p(t)$ for each of the following differential equations. Do not solve for the resulting constants.

1. $y'' - 5y' + 7y = 4e^{3t}$

► **Solution.** This is Case (1) with $\alpha = 3$. Since $p(s) = s^2 - 5s + 7$ and $p(3) = 1 \neq 0$, $\varphi_p(t) = Ce^{3t}$. ◀

2. $y'' - 5y' + 7y = 2t - t^3$

► **Solution.** This is Case (1) with $\alpha = 0$. Since $p(0) = 7 \neq 0$, it follows that

$$\varphi_p(t) = C_0 + C_1t + C_2t^2 + C_3t^3.$$

◀

3. $y'' - 5y' = 2t - t^3$

► **Solution.** This is again Case (1) with $\alpha = 0$. Since $p(0) = 0$, it follows that

$$\varphi_p(t) = C_0t + C_1t^2 + C_2t^3 + C_3t^4.$$

◀

4. $y'' + y' - 6y = 2e^t - e^{2t}$

► **Solution.** Since $p(s) = (s - 2)(s + 3)$, we have that 2 is a simple root of $p(s)$. Hence, using both parts (a) and (b) of Case (1), it follows that $\varphi_p(t) = C_1e^t + C_2te^{2t}$.

◀

5. $y'' + 4y = 5e^{-3t} \sin 2t$

► **Solution.** Since $\alpha + i\beta = -3 + 2i$ is not a root of $p(s) = s^2 + 4$, Case (2) shows

$$\varphi_p(t) = C_1e^{-3t} \sin 2t + C_2e^{-3t} \cos 2t.$$

◀

6. $y'' + 6y' + 13y = 5e^{-3t} \sin 2t$

► **Solution.** Since $\alpha + i\beta = -3 + 2i$ is a root of $p(s) = s^2 + 6s + 13 = (s + 3)^2 + 4$, Case (2) shows

$$\varphi_p(t) = C_1te^{-3t} \sin 2t + C_2te^{-3t} \cos 2t.$$

◀

Exercises

Find the general solution of each of the following differential equations.

1. $y'' + 3y' + 2y = 4$
2. $y'' + 3y' + 2y = 12e^t$
3. $y'' + 3y' + 2y = \sin t$
4. $y'' + 3y' + 2y = \cos t$
5. $y'' + 3y' + 2y = 8 + 6e^t + 2\sin t$
6. $y'' - 3y' - 4y = 6e^t$
7. $y'' - 3y' - 4y = 5e^{4t}$
8. $y'' - 4y' + 3y = 20\cos t$
9. $y'' - 4y' + 3y = 2\cos t + 4\sin t$
10. $y'' - 4y = 8e^{2t} - 12$
11. $y'' - 3y' + 2y = 2t^3 - 9t^2 + 6t$
12. $y'' - 3y' + 2y = 2t^2 + 1$
13. $y'' + 4y = 5e^t - 4t$
14. $y'' + 4y = 5e^t - 4t^2$
15. $y'' + y' + y = t^2$
16. $y'' - 2y' - 8y = 9te^t + 10e^{-t}$
17. $y'' - 3y' = 2e^{2t}\sin t$
18. $y'' + y' = t^2 + 2t$
19. $y'' + y' = t + \sin 2t$
20. $y'' + y = \cos t$
21. $y'' + y = 4t\sin t$
22. $y'' - 3y' - 4y = 16t - 50\cos 2t$
23. $y'' + 4y' + 3y = 15e^{2t} + e^{-t}$
24. $y'' - y' - 2y = 6t + 6e^{-t}$

25. $y'' + y = \sin^2 t$ *Hint:* $\sin^2 t = \frac{1}{2} - \frac{1}{2} \cos 2t$

26. $y'' - 4y' + 4y = e^{2t}$

Solve each of the following initial value problems.

27. $y'' - 5y' - 6y = e^{3t}$, $y(0) = 2$, $y'(0) = 1$.

28. $y'' + 2y' + 5y = 8e^{-t}$, $y(0) = 0$, $y'(0) = 8$.

29. $y'' + y = 10e^{2t}$, $y(0) = 0$, $y'(0) = 0$.

30. $y'' - 4y = 2 - 8t$, $y(0) = 0$, $y'(0) = 5$.

31. $y'' - y' - 2y = 5 \sin t$, $y(0) = 1$, $y'(0) = -1$.

32. $y'' + 9y = 8 \cos t$, $y(\pi/2) = -1$, $y'(\pi/2) = 1$.

33. $y'' - 5y' + 6y = e^t(2t - 3)$, $y(0) = 1$, $y'(0) = 3$.

34. $y'' - 3y' + 2y = e^{-t}$, $y(0) = 1$, $y'(0) = -1$.

3.6 Variation of Parameters

Let $\mathbf{L} = \mathbf{D}^2 + a\mathbf{D} + b$, where a and b are continuous functions on an interval I . In this section we address the issue of finding a particular solution to a nonhomogeneous linear differential equation $\mathbf{L}(y) = f$, where f is continuous on I . It is a pleasant and remarkable feature of linear differential equations that the homogeneous solutions can be used decisively to find a particular solution. The procedure we use is called **variation of parameters**.

Suppose $\{\varphi_1, \varphi_2\}$ is a fundamental set for $\mathbf{L}(y) = 0$. We know then that all solutions of the homogeneous equation $\mathbf{L}(y) = 0$ are of the form $c_1\varphi_1 + c_2\varphi_2$. To find a particular solution φ_p to $\mathbf{L}(y) = f$ the method of variation of parameters makes two assumptions. First, the parameters c_1 and c_2 are allowed to vary. We thus replace the constants c_1 and c_2 by functions $u_1(t)$ and $u_2(t)$, and assumes that the particular solution φ_p , takes the form

$$\varphi_p(t) = u_1(t)\varphi_1(t) + u_2(t)\varphi_2(t). \quad (1)$$

The second assumption is

$$u_1'(t)\varphi_1(t) + u_2'(t)\varphi_2(t) = 0. \quad (2)$$