Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper. A table of Laplace transforms and a short table of integrals are appended to the exam.

1. [**17 Points**] Solve the initial value problem: $y' = \dfrac{3t^2}{2(y-1)}, \quad y(0) = 0.$

   ▶ **Solution.** This equation is separable. There is one equilibrium solution: $y = 1$. This does not satisfy the initial condition so it is not the required solution. If $y \neq 1$, separate variables to get $2(y-1)y' = 3t^2$, which in differential form is

   $$2(y-1)\,dy = 3t^2\,dt$$

   and integration then gives $(y-1)^2 = t^3 + C$. The initial condition $y(0) = 0$ gives $(-1)^2 = C$ so $C = 1$, and $(y-1)^2 = t^3 + 1$. Solving for y gives $y = 1 \pm \sqrt{t^3+1}$. The initial condition $y(0) = 0$ requires that the minus sign be used. Thus,

   $$\boxed{y = 1 - \sqrt{t^3+1}.}$$

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2. [**17 Points**] Solve the initial value problem: $y' + 2y = 3e^{2t} + 4e^{-2t} \quad y(0) = -1.$

   ▶ **Solution.** This equation is linear with coefficient function $p(t) = 2$ so that an integrating factor is given by $\mu(t) = e^{\int 2\,dt} = e^{2t}$. Multiplication of the differential equation by the integrating factor gives

   $$e^{2t}y' + 2e^{2t}y = 3e^{4t} + 4,$$

   and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

   $$\frac{d}{dt}(e^{2t}y) = 3e^{4t} + 4.$$

   Integration then gives

   $$e^{2t}y = \frac{3}{4}e^{4t} + 4t + C,$$

   where C is an integration constant. Multiplying through by e^{-2t} gives

   $$y = \frac{3}{4}e^{2t} + 4te^{-2t} + Ce^{-2t}.$$

   The initial condition gives $-1 = y(0) = \frac{3}{4} + C$ so $C = -\frac{7}{4}$ and thus

   $$\boxed{y = \frac{3}{4}e^{2t} + 4te^{-2t} - \frac{7}{4}e^{-2t}.}$$

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3. **[17 Points]** Solve the initial value problem: $y' + \dfrac{2t}{1+t^2}y = 1, \quad y(0) = 0.$

▶ **Solution.** This equation is linear with coefficient function $p(t) = 2t/(1+t^2)$, so that an integrating factor is $\mu(t) = e^{\int (2t/(1+t^2))\,dt} = e^{\ln(1+t^2)} = 1+t^2$. Multiplication of the differential equation by the integrating factor gives

$$(1+t^2)y' + 2ty = 1+t^2,$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}((1+t^2)y) = 1+t^2.$$

Integration then gives

$$(1+t^2)y = t + t^3/3 + C,$$

where C is an integration constant. Multiplying through by $1+t^2$ gives

$$y = \frac{t + t^3/3 + C}{1+t^2}.$$

The initial condition $y(0) = 0$ implies $0 = y(0) = \frac{C}{1}$ so that $C = 0$. Hence

$$\boxed{y = \frac{3t+t^3}{3(1+t^2)}.}$$

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4. **[17 Points]** Find the general solution of: $2ty^2 - 3 + (2t^2y + 4)y' = 0$.

▶ **Solution.** Letting $M(t,\, y) = 2ty^2 - 3$ and $N(t,\, y) = 2t^2y + 4$, the equation can be written in the form $M + Ny' = 0$, and since

$$\frac{\partial M}{\partial y} = 4ty = \frac{\partial N}{\partial t},$$

it follows that the equation is exact. Hence, we solve it by looking for a potential function $V(t,\, y)$ such that $M = \partial V/\partial t$ and $N = \partial V/\partial y$. Find V by integration:

$$V(t,\, y) = \int \frac{\partial V}{\partial t}\,dt = \int M\,dt = \int (2ty^2 - 3)\,dt = t^2y^2 - 3t + \varphi(y),$$

where $\varphi(y)$ is a function depending only on y. Now use

$$2t^2y + 4 = N(t,\, y) = \frac{\partial V}{\partial y} = 2t^2y + \frac{d}{dy}\varphi(y),$$

which implies that $\dfrac{d}{dy}\varphi(y) = 4$ so that $\varphi(y) = 4y$. Hence, the solutions to the differential equation are given by the implicit solutions of $V(t,\, y) = C$, i.e.,

$$\boxed{t^2y^2 - 3t + 4y = C,}$$

where C is a constant. ◀

5. **[15 Points]** Compute the Laplace transform of each of the following functions. You may use the attached tables, but be sure to identify which formulas you are using by citing the number(s) or name of the formula in the table.

   (a) $f(t) = 3t^4 - 5te^{-6t} + 7$

   ▶ **Solution.** From Linearity and Formula 2, with $n = 3$ and $n = 0$, and Formula 4 with $n = 1$, $a = -6$,

$$\boxed{F(s) = \frac{3 \cdot 4!}{s^5} - 5\frac{1}{(s+6)^2} + \frac{7}{s} = \frac{72}{s^5} - \frac{5}{(s+6)^2} + \frac{7}{s}.}$$

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   (b) $g(t) = (t-3)^2 e^{4t} + 5e^{-2t}\cos 3t$

   ▶ **Solution.** First expand $(t-3)^2 = t^2 - 6t + 9$ to write $g(t) = t^2e^{4t} - 6te^{4t} + 9e^{4t} + 5e^{-2t}\cos 3t$. Now use Linearity and Formula 4 with $n = 2$, $n = 1$ and $n = 0$ with $a = 4$, plus Formula 7 with $a = -2$, $b = 3$ to get

$$\boxed{G(s) = \frac{2}{(s-4)^3} - \frac{6}{(s-4)^2} + \frac{9}{s-4} + \frac{5(s+2)}{(s+2)^2+9}.}$$

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   (c) $h(t) = 3t\sin 5t$

   ▶ **Solution.** Use the Transform Derivative Principle and Formula 6 to get

$$\begin{aligned} H(s) &= -3\frac{d}{ds}\left(\frac{5}{s^2+25}\right) \\ &= -3 \cdot 5\left(\frac{-2s}{(s^2+25)^2}\right) \\ &= \frac{30s}{(s^2+25)^2}. \end{aligned}$$

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6. **[17 Points]** A 400 gallon tank is initially full of brine which contains 60 pounds of salt. A solution containing 0.5 pounds of salt per gallon enters the tank at a rate of 6 gallons per minute. A drain is opened at the bottom of the tank through which the well stirred solution leaves the tank at the same rate of 6 gallons per minute. Let $y(t)$ denote the amount of salt (in pounds) which is in the tank at time t.

   (a) What is $y(0)$? $\boxed{y(0) = 60}$

   (b) Write the differential equation that $y(t)$ must satisfy.

▶ **Solution.** The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0.5 lb/gal × 6 gal/min, i.e., 3 lb/min. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 6.$$

Since mixture is entering and leaving at the same volume rate of 6 gal/min, the volume of mixture in the tank is constant. Thus $V(t) = 400$. Hence $y(t)$ satisfies the equation

$$y' = 3 - \frac{6}{400}y,$$

so that the initial value problem satisfied by $y(t)$ is

$$\boxed{y' + \frac{3}{200}y = 3, \qquad y(0) = 60.}$$

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(c) Solve the differential equation to find $y(t)$.

▶ **Solution.** This is a linear differential equation with integrating factor $\mu(t) = e^{3t/200}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(e^{3t/200}y\right) = 3e^{3t/200}.$$

Integration of this equation gives

$$e^{3t/200}y = 200e^{3t/200} + C,$$

where C is an integration constant. Dividing by $e^{3t/200}$ gives

$$y = 200 + Ce^{-3t/200},$$

and the initial condition $y(0) = 60$ gives a value of $C = -140$. Hence, the amount of salt at time t is

$$\boxed{y(t) = 200 - 140e^{-3t/200}.}$$

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(d) How much salt is in the tank after 1 hour?

▶ **Solution.** This is obtained by taking $t = 60$ (minutes) in the previous equation:

$$\boxed{y(60) = 200 - 140e^{-180/200} \approx 143.08 \text{ lb}}$$

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Laplace Transform Tables

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\{f(bt)\}(s)$	$=$	$\dfrac{1}{b}\mathcal{L}\{f(t)\}(s/b).$

Some Integral Formulas

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ (if $n \neq -1$)
2. $\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\int \frac{1}{a+bx}\,dx = \frac{1}{b}\ln|a+bx| + C \quad (b \neq 0)$
4. $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C \quad (a > 0)$
5. $\int \frac{1}{x(a+bx)}\,dx = \frac{1}{a}\ln\left|\frac{x}{a+bx}\right| + C$
6. $\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
7. $\int \ln x\,dx = x\ln x - x + C$
8. $\int xe^{ax}\,dx = \frac{xe^{ax}}{a} - \frac{e^{ax}}{a^2} + C$
9. $\int \cos ax\,dx = \frac{\sin ax}{a} + C$
10. $\int \sin ax\,dx = -\frac{\cos ax}{a} + C$