Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, a table of convolutions, and the statement of the main partial fraction decomposition theorem have been appended to the exam.

1. **[20 Points]** Compute the inverse Laplace transform of each of the following rational functions.

   (a) $F(s) = \dfrac{8s+16}{(s-2)^2(s^2+4)}$

   ▶ **Solution.** $\dfrac{8s+16}{(s-2)^2(s^+4)} = \dfrac{A_1}{(s-2)^2} + \dfrac{p_1(s)}{(s-2)(s^2+4)}$ where $A_1 = \left.\dfrac{8s+16}{s^2+4}\right|_{s=2} = 32/8 = 4$ and

$$p_1(s) = \frac{(8s+16) - 4(s^2+4)}{s-2} = \frac{8s-4s^2}{s-2} = -4s.$$

   Thus,

$$\frac{8s+16}{(s-2)^2(s^2+4)} = \frac{4}{(s-2)^2} + \frac{-4s}{(s-2)(s^2+4)}$$

   and

$$\frac{-4s}{(s-2)(s^2+4)} = \frac{A_2}{s-2} + \frac{p_2(s)}{s^2+4}$$

   where $A_2 = \left.\dfrac{-4s}{s^2+4}\right|_{s=2} = -8/8 = -1$ and

$$p_2(s) = \frac{-4s - (-1)(s^2+4)}{s-2} = \frac{s^2-4s+4}{s-2} = s-2.$$

   Therefore,

$$\frac{8s+16}{(s-2)^2(s^2+4)} = \frac{4}{(s-2)^2} + \frac{-1}{s-2} + \frac{s-2}{s^2+4}$$

   and hence

$$\boxed{\mathcal{L}^{-1}\left\{\frac{8s+16}{(s-2)^2(s^2+4)}\right\} = 4te^{2t} - e^{2t} + \cos 2t - \sin 2t.}$$

◀

   (b) $G(s) = \dfrac{3s-2}{s^2+4s+9}$

   ▶ **Solution.**

$$\begin{aligned} G(s) &= \frac{3s-2}{s^2+4s+9} = \frac{3s-2}{(s+2)^2+5} = \frac{3(s+2)-8}{(s+2)^2+5} \\ &= \frac{3(s+2)}{(s+2)^2+5} - \frac{8}{(s+2)^2+5}. \end{aligned}$$

Therefore,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3s-2}{s^2+4s+9}\right\} = 3e^{-2t}\cos\sqrt{5}t - \frac{8}{\sqrt{5}}e^{-2t}\sin\sqrt{5}t.}$$

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2. **[14 Points]**

(a) Find the Laplace transform $Y(s)$ of the solution $y(t)$ of the initial value problem

$$y'' + 4y = 3\sin t, \qquad y(0) = 0,\ y'(0) = 0.$$

▶ **Solution.** Applying the Laplace transform to the differential equation gives

$$s^2Y(s) + 4Y(s) = \frac{3}{s^2+1}.$$

This implies that

$$(s^2+4)Y(s) = \frac{3}{s^2+1},$$

and solving for $Y(s)$ gives

$$\boxed{Y(s) = \frac{3}{(s^2+4)(s^2+1)}}$$

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(b) Using your answer from part (a), find $y(t)$.

▶ **Solution.** Compute the partial fraction decomposition of $Y(s)$, starting with s^2+1 which has roots $\pm i$:

$$Y(s) = \frac{3}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+1} + \frac{p_1(s)}{s^2+4}$$

where $Ai + B = \left.\frac{3}{s^2+4}\right|_{s=i} = 3/(-1+4) = 1$. Thus, $A = 0$ and $B = 1$, and

$$p_1(s) = \frac{3-(1)(s^2+4)}{s^2+1} = \frac{-s^2-1}{s^2+1} = -1.$$

Thus

$$Y(s) = \frac{1}{s^2+1} - \frac{1}{s^2+4},$$

and hence,

$$\boxed{y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+4}\right\} = \sin t - \frac{1}{2}\sin 2t.}$$

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3. [**10 Points**] Find the standard basis $\mathcal{B}_q$ of $\mathcal{E}_q$ for $q(s) = s^2(s-2)^3(s^2+9)^2$.

   ▶ **Solution.** $\boxed{\mathcal{B}_q = \left\{1,\, t,\, e^{2t},\, te^{2t},\, t^2e^{2t},\, \cos 3t,\, \sin 3t,\, t\cos 3t,\, t\sin 3t\right\}}$ ◀

4. [**15 Points**] Solve the initial value problem: $4y'' + 4y' + y = 0, \quad y(0) = 2,\ y'(0) = 1$.

   ▶ **Solution.** The characteristic polynomial is $q(s) = 4s^2 + 4s + 1 = (2s+1)^2$, which has a single root $-1/2$ of multiplicity 2. Thus, $\mathcal{B}_q = \left\{e^{-t/2},\, te^{-t/2}\right\}$ so that the general solution is $y = c_1e^{-t/2} + c_2te^{-t/2}$. Since $y' = (-\frac{c_1}{2} + c_2)e^{-t/2} - \frac{1}{2}c_2te^{-t/2}$, the initial conditions give,

$$\begin{aligned} 2 = y(0) &= c_1 \\ 1 = y'(0) &= -\frac{1}{2}c_1 + c_2. \end{aligned}$$

   The first equation gives $c_1 = 2$, and the second gives $c_2 = 2$. Hence,

$$\boxed{y(t) = 2e^{-t/2} + 2te^{-t/2}.}$$

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5. [**13 Points**] Find the general solution of $2y'' + 7y' + 6y = 0$.

   ▶ **Solution.** The characteristic polynomial is $q(s) = 2s^2 + 7s + 6 = (2s+3)(s+2)$, which has a roots $-3/2$ and -2. Hence, the general solution is

$$\boxed{y(t) = c_1e^{-3t/2} + c_2e^{-2t}.}$$

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6. [**13 Points**] Find the general solution of $y'' + 8y' + 25y = 0$.

   ▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 8s + 25 = (s+4)^2 + 9$ which has roots $-4 \pm 3i$. Hence, the general solution is

$$\boxed{y(t) = c_1e^{-4t}\cos 3t + c_2e^{-4t}\sin 3t.}$$

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7. [**15 Points**] Find the general solution of the following differential equation using the method of undetermined coefficients:

$$y'' - 3y' - 10y = 5e^{-2t}.$$

▶ **Solution.** The characteristic polynomial of the associated homogeneous equation is $q(s) = s^2 - 3s - 10 = (s+2)(s-5)$. $F(s) = \mathcal{L}\left\{5e^{-2t}\right\} = \dfrac{6}{s+2}$ which has denominator $v(s) = s + 2$. Thus, $q(s)v(s) = (s+2)^2(s-5)$, and

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q - \left\{e^{-2t},\, te^{-2t},\, e^{5t}\right\} \setminus \left\{e^{-2t},\, e^{5t}\right\} = \left\{te^{-2t}\right\}.$$

Thus, the particular solution has the form $y_p = Ate^{-2t}$. Compute the derivatives $y_p' = A(1-2t)e^{-2t}$, and $y_p'' = A(-4+4t)e^{-2t}$ and substitute in the differential equation to get

$$5e^{-2t} = y_p'' - 3y_p' - 10y_p = A(-4+4t)e^{-2t} - 3A(1-2t)e^{-2t} - 10Ate^{-2t} = -7Ae^{-2t}.$$

Thus, $A = -5/7$ so $y_p(t) = (-5/7)te^{-2t}$, and the general solution is

$$\boxed{y_g(t) = y_h(t) + y_p(t) = c_1e^{-2t} + c_2e^{5t} - \frac{5}{7}te^{-2t}.}$$

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Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\frac{1}{s}$
2.	t^n	$\to$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\frac{b}{(s-a)^2+b^2}$

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\frac{d}{ds}F(s)$
The Dilation Principle	$\mathcal{L}\{f(bt)\}(s)$	$=$	$\frac{1}{b}\mathcal{L}\{f(t)\}(s/b)$
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Table of Convolutions

$f(t)$	$g(t)$	$(f * g)(t)$
1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
t^m	t^n	$\dfrac{m!n!}{(m+n+1)!}t^{m+n+1}$
t	$\sin at$	$\dfrac{at-\sin at}{a^2}$
t^2	$\sin at$	$\dfrac{2}{a^3}(\cos at-(1-\frac{a^2t^2}{2}))$
t	$\cos at$	$\dfrac{1-\cos at}{a^2}$
t^2	$\cos at$	$\dfrac{2}{a^3}(at-\sin at)$
t	e^{at}	$\dfrac{e^{at}-(1+at)}{a^2}$
t^2	e^{at}	$\dfrac{2}{a^3}(e^{at}-(a+at+\frac{a^2t^2}{2}))$
e^{at}	e^{bt}	$\dfrac{1}{b-a}(e^{bt}-e^{at}) \quad a\neq b$
e^{at}	e^{at}	te^{at}
e^{at}	$\sin bt$	$\dfrac{1}{a^2+b^2}(be^{at}-b\cos bt-a\sin bt)$
e^{at}	$\cos bt$	$\dfrac{1}{a^2+b^2}(ae^{at}-a\cos bt+b\sin bt)$
$\sin at$	$\sin bt$	$\dfrac{1}{b^2-a^2}(b\sin at-a\sin bt) \quad a\neq b$
$\sin at$	$\sin at$	$\dfrac{1}{2a}(\sin at-at\cos at)$
$\sin at$	$\cos bt$	$\dfrac{1}{b^2-a^2}(a\cos at-a\cos bt) \quad a\neq b$
$\sin at$	$\cos at$	$\dfrac{1}{2}t\sin at$
$\cos at$	$\cos bt$	$\dfrac{1}{a^2-b^2}(a\sin at-b\sin bt) \quad a\neq b$
$\cos at$	$\cos at$	$\dfrac{1}{2a}(at\cos at+\sin at)$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$

Reduction of order formulas

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^{k+1}}\right\} &= \frac{-t}{2kb^2}\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^k}\right\} + \frac{2k-1}{2kb^2}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\} \\
\mathcal{L}^{-1}\left\{\frac{s}{(s^2+b^2)^{k+1}}\right\} &= \frac{t}{2k}\mathcal{L}^{-1}\left\{\frac{1}{(s^2+b^2)^k}\right\}
\end{aligned}$$