Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam.

1. [**15 Points**] Each of the following matrices is in reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For each of these matrices, write down all solutions to the linear system of equations that has the given matrix as augmented matrix.

▶ **Solution.** For A: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$.

For B: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2\alpha + 3 \\ \alpha \\ 4 \end{bmatrix}$ where $x_2 = \alpha$ is arbitrary.

For C: The last row corresponds to the equation $0 = 1$. Thus the system has no solutions. ◀

2. [**20 Points**] Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) Compute A^{-1}.

▶ **Solution.**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

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(b) Using your answer to part (a), solve the linear system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$.

▶ **Solution.** $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$. ◀

3. [**15 Points**] Let $C = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 2 & 4 & 6 & 2 \\ 1 & -1 & 6 & 0 \\ 0 & 0 & 1 & 5 \end{bmatrix}$. Compute $\det C$.

▶ **Solution.** Use cofactor expansion along the first row to get

$$\begin{aligned} \det C &= 2\det\begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} - (1)\det\begin{bmatrix} 2 & 4 & 6 \\ 1 & -1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \\ &= 2\cdot 5\det\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - (1)\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} = 9\begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \\ &= 9(-2-4) = 9(-6) = -54. \end{aligned}$$

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4. [**15 Points**] If $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$, compute the matrix exponential e^{At}.

▶ **Solution.** $sI - A = \begin{bmatrix} 2 & -2 \\ -1 & s-1 \end{bmatrix}$ so $\det(sI - A) = s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1)$. Hence,

$$\begin{aligned} (sI - A)^{-1} &= \frac{1}{(s-2)(s+1)}\begin{bmatrix} s-1 & 2 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-2)(s+1)} & \frac{2}{(s-2)(s+1)} \\ \frac{1}{(s-2)(s+1)} & \frac{s}{(s-2)(s+1)} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1/3}{s-2} + \frac{2/3}{s+1} & \frac{2/3}{s-2} - \frac{2/3}{s+1} \\ \frac{1/3}{s-2} - \frac{1/3}{s+1} & \frac{2/3}{s-2} + \frac{1/3}{s+1} \end{bmatrix}. \end{aligned}$$

Then,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \frac{1}{3}\begin{bmatrix} e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t} \\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t} \end{bmatrix}.$$

Alternatively, you can use Fulmer's method:

Since $c_A(s) = (s-2)(s+1)$, the standard basis for $\mathcal{E}_{c_A(s)}$ is $\{e^{2t},\, e^{-t}\}$ so $e^{At} = M_1e^{2t} + M_2e^{-t}$. Differentiating gives $Ae^{At} = 2M_1e^{2t} - M_1e^{-t}$. Evaluating at $t = 0$ gives the system of equations

$$\begin{aligned} I &= M_1 + M_2 \\ A &= 2M_1 - M_2. \end{aligned}$$

Solving for M_1 and M_2 gives $M_1 = \frac{1}{3}(I + A)$ and $M_2 = I - M_1 = \frac{1}{3}(2I - A)$. Thus, $M_1 = \frac{1}{3}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$ and $M_2 = \frac{1}{3}\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix}$ so that

$$\begin{aligned} e^{At} &= \frac{1}{3}\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix} e^{2t} + \frac{1}{3}\begin{bmatrix}2 & -2\\ -1 & 1\end{bmatrix} e^{-t} \\ &= \frac{1}{3}\begin{bmatrix}e^{2t} + 2e^{-t} & 2e^{2t} - 2e^{-t}\\ e^{2t} - e^{-t} & 2e^{2t} + e^{-t}\end{bmatrix}. \end{aligned}$$

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5. [**20 Points**] Find the general solution of the system of differential equations

$$\begin{aligned} y_1' &= 3y_1 - 6y_2 \\ y_2' &= 3y_1 - 3y_2 \end{aligned}$$

You may assume that the characteristic polynomial of the matrix $A = \begin{bmatrix}3 & -6\\ 3 & -3\end{bmatrix}$ is $c_A(s) = s^2 + 9$.

▶ **Solution.** The general solution is $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$ where $\mathbf{y}(0) = \begin{bmatrix}c_1\\ c_2\end{bmatrix}$ is arbitrary. Compute e^{At} by Fulmer's method. Since $c_A(s) = s^2 + 9$ the standard basis of $\mathcal{E}_{c_A(s)}$ is $\{\cos 3t, \sin 3t\}$, so that $e^{At} = M_1 \cos 3t + M_2 \sin 3t$. Differentiating gives $Ae^{At} = -3M_1 \sin 3t + 3M_2 \cos 3t$. Evaluating at $t = 0$ gives $I = M_1$, $A = 3M_2$. Thus, $M_1 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $M_2 = \frac{1}{3}A = \begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}$, and hence

$$e^{At} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\cos 3t + \begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}\sin 3t = \begin{bmatrix}\cos 3t + \sin 3t & -2\sin 3t\\ \sin 3t & \cos 3t - \sin 3t\end{bmatrix}.$$

Therefore, the general solution is

$$\begin{aligned} \mathbf{y}(t) = \begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} &= e^{At}\mathbf{y}(0) = e^{At}\begin{bmatrix}c_1\\ c_2\end{bmatrix} \\ &= \begin{bmatrix}\cos 3t + \sin 3t & -2\sin 3t\\ \sin 3t & \cos 3t - \sin 3t\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix} = \begin{bmatrix}c_1(\cos 3t + \sin 3t) - 2c_2 \sin 3t\\ c_1 \sin 3t + c_2(\cos 3t - \sin 3t\end{bmatrix} \\ &= \begin{bmatrix}c_1 \cos 3t + (c_1 - 2c_2)\sin 3t\\ c_2 \cos 3t + (c_1 - c_2)\sin 3t\end{bmatrix}. \end{aligned}$$

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6. [**20 Points**] If $A = \begin{bmatrix}5 & 2\\ -2 & 1\end{bmatrix}$, solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix}2\\ -1\end{bmatrix}$. You may assume that the characteristic polynomial of A is $c_A(s) = (s-3)^2$.

▶ **Solution.** Compute e^{At}, using either the Laplace transform method or Fulmer's method. We will use the Laplace transform method. First compute the resolvent matrix $(sI-A)^{-1}$. Since $\det(sI-A) = (s-3)^2$,

$$
\begin{aligned}
(sI-A)^{-1} &= \frac{1}{(s-3)^2}\begin{bmatrix} s-1 & 2 \\ -2 & s-5 \end{bmatrix} = \begin{bmatrix} \frac{s-1}{(s-3)^2} & \frac{2}{(s-3)^2} \\ \frac{-2}{(s-3)^2} & \frac{s-5}{(s-3)^2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{(s-3)+2}{(s-3)^2} & \frac{2}{(s-3)^2} \\ \frac{-2}{(s-3)^2} & \frac{(s-3)-2}{(s-3)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s-3} + \frac{2}{(s-3)^2} & \frac{2}{(s-3)^2} \\ \frac{-2}{(s-3)^2} & \frac{1}{s-3} - \frac{2}{(s-3)^2} \end{bmatrix}
\end{aligned}
$$

Thus

$$
e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} e^{3t} + 2te^{3t} & 2te^{3t} \\ -2te^{3t} & e^{3t} - 2te^{3t} \end{bmatrix},
$$

and the solution of the initial value problem is then

$$
\begin{aligned}
\mathbf{y}(t) = e^{At}\begin{bmatrix} 2 \\ -1 \end{bmatrix} &= \begin{bmatrix} e^{3t} + 2te^{3t} & -2te^{3t} \\ 2te^{3t} & e^{3t} - 2te^{3t} \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} 2(e^{3t} + 2te^{3t}) - 2te^{3t} \\ -4te^{3t} - (e^{3t} - 2te^{3t}) \end{bmatrix} \\
&= \begin{bmatrix} 2e^{3t} + 2te^{3t} \\ -e^{3t} - 2te^{3t} \end{bmatrix}.
\end{aligned}
$$

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Laplace Transform Table

	$f(t)$	$\rightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\rightarrow$	$\frac{1}{s}$
2.	t^n	$\rightarrow$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\rightarrow$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\rightarrow$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\rightarrow$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\rightarrow$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\rightarrow$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\rightarrow$	$\frac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\rightarrow$	$\frac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\rightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\frac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.