Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms is attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $ty' - 3y = t^3, \quad y(1) = 10.$
2. [**12 Points**] $(t^2 + 1)y' + 2ty^2 = 0, \quad y(0) = 2.$
3. [**10 Points**] $y'' + 6y' + 25y = 0.$
4. [**10 Points**] $9y'' + 12y' + 4y = 0.$
5. [**12 Points**] $2y'' - 5y' - 25y = 0, \quad y(0) = 3,\ y'(0) = 0.$
6. [**12 Points**] $y'' - y' - 2y = 4e^{2t}.$
7. [**12 Points**] Find a 2π-periodic solution of the differential equation
$$y'' + 3y = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt.$$
8. [**12 Points**] Find the general solution of the differential equation
$$ty'' - y' = 3t^2 - 1,$$
given the fact that two solutions of the associated homogeneous equation are $y_1(t) = 1$ and $y_2(t) = t^2$.
9. [**12 Points**] Let $f(t)$ be the following function:
$$f(t) = \begin{cases} \sin t & \text{if } 0 \le t < \pi, \\ \cos t & \text{if } t \ge \pi. \end{cases}$$
   (a) Sketch the graph of $f(t)$ on the interval $[0,\ 3\pi]$.
   (b) Compute the Laplace transform of $f(t)$.
10. [**10 Points**] Compute the following inverse Laplace transform:
$$\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}e^{-3s}\right\}$$

11. [**12 Points**] Let $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$.

    (a) Compute e^{At}.

    (b) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

12. [**12 Points**]Let $f(t)$ be the periodic function of period 2π that is defined on the interval $(-\pi, \pi]$ by

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t \le 0, \\ t & \text{if } 0 < t \le \pi. \end{cases}$$

    (a) Sketch the graph of $f(t)$ on the interval $[-3\pi, 3\pi]$.

    (b) Compute the Fourier series of $f(t)$. The following integration formulas may be of use:

$$\int t \sin at \, dt = \frac{1}{a^2} \sin at - \frac{t}{a} \cos at \qquad \int t \cos at \, dt = \frac{1}{a^2} \cos at + \frac{t}{a} \sin at.$$

    (c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(0)$, $g(\pi)$, and $g(5\pi/2)$.

13. [**12 Points**] A 10-liter jug of water is contaminated with 100 grams of salt. Assume that the jug is always well-mixed. Starting at time $t = 0$, pure water begins flowing into the jug at a rate of 2 liters per minute. Well-mixed salty water flows out of the jug at the rate of 3 liters per minute. This means that the entire jug will be empty at time $t = 10$ minutes. How much salt will be in the jug at time $t = 5$ minutes?

Laplace Transform Table

	$f(t)$	$\rightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\rightarrow$	$\dfrac{1}{s}$
2.	t^n	$\rightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\rightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\rightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\rightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\rightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\rightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\rightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\rightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\rightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.