Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms is attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $ty' - 3y = t^3, \quad y(1) = 10.$

   ▶ **Solution.** This equation is linear. To find an integrating factor first divide by t to put it in standard form $y' - \dfrac{3}{t}y = t^2$. Then $p(t) = -3/t$ so that an integrating factor is $\mu(t) = e^{\int p(t)\,dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$. Multiplying the equation by t^{-3} gives

   $$t^{-3}y' - \frac{3}{t}t^{-3}y = \frac{1}{t}.$$

   Thus,

   $$\frac{d}{dt}\left(t^{-3}y\right) = \frac{1}{t},$$

   and integrating gives

   $$t^{-3}y = \int \frac{1}{t}\,dt = \ln t + C.$$

   Hence, solving for y gives $y = t^3\ln t + Ct^3$. Now apply the initial conditions $y(0) = 10$ to get $10 = y(0) = C$. Therefore,

   $$\boxed{y(t) = t^3\ln t + 10t^3.}$$

   ◀

2. [**12 Points**] $(t^2+1)y' + 2ty^2 = 0, \quad y(0) = 2.$

   ▶ **Solution.** This equation is separable. Separate the variables to get $\dfrac{y'}{y^2} = -\dfrac{2t}{t^2+1}$. Write in differential form and integrate to get:

   $$\int \frac{dy}{y^2} = \int -\frac{2t}{t^2+1}\,dt.$$

   Integrating gives $-1/y = -\ln(t^2+1) + C$. Multiplying by -1 and taking the reciprocal of both sides gives

   $$y = \frac{1}{\ln(t^2+1) - C}.$$

Apply the initial condition $y(0) = 2$ to get

$$2 = y(0) = \frac{1}{\ln 1 - C} = \frac{1}{-C}.$$

Thus, $C = -1/2$ and hence, after multiplying the numerator and denominator by 2 we get

$$\boxed{y(t) = \frac{2}{2\ln(t^2+1)+1}.}$$

◀

3. [**10 Points**] $y'' + 6y' + 25y = 0$.

   ▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 6s + 25 = (s+3)^2 + 16$, which has roots $-3 \pm 4i$. Thus, the solutions are given by

   $$\boxed{y(t) = c_1 e^{-3t}\cos 4t + c_2 e^{-3t}\sin 4t.}$$

   ◀

4. [**10 Points**] $9y'' + 12y' + 4y = 0$. The characteristic polynomial is $q(s) = 9s^2 + 12s + 4 = (3s+2)^2$, which has a single root $-2/3$ of multiplicity 2. Thus, the solutions are given by

   $$\boxed{y(t) = c_1 e^{-2t/3} + c_2 t e^{-2s/3}.}$$

5. [**12 Points**] $2y'' - 5y' - 25y = 0, \quad y(0) = 3,\ y'(0) = 0.$

   ▶ **Solution.** The characteristic polynomial is $q(s) = 2s^2 - 5s - 25 = (2s+5)(s-5)$, which has roots $-5/2$ and 5. Thus, the general solution of the differential equation is $y(t) = c_1 e^{-5t/2} + c_2 e^{5t}$. The initial conditions give the equations for c_1, c_2:

   $$\begin{aligned} 3 &= y(0) = c_1 + c_2 \\ 0 &= y'(0) = -\frac{5}{2}c_1 + 5c_2. \end{aligned}$$

   Solve these equations to get $c_1 = 2$, $c_2 = 1$. Thus, the solution of the initial value problem is

   $$\boxed{y(t) = 2e^{-5t/2} + e^{5t}.}$$

   ◀

6. [**12 Points**] $y'' - y' - 2y = 4e^{2t}$.

▶ **Solution.** Use the method of undetermined coefficients. The characteristic polynomial of the associated homogeneous equation is $q(s) = s^2 - s - 2 = (s+1)(s-2)$. The roots are -1 and 2 so that the standard basis is $\mathcal{B}_q = \{e^{-t}, e^2t\}$. The Laplace transform of the right hand side is $\mathcal{L}\{4e^{2t}\} = 4/(s-2)$, which has denominator $v(s) = s-2$. Thus, $q(s)v(s) = (s+1)(s-2)^2$ and

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{-t}, e^{2t}, te^{2t}\right\} \setminus \left\{e^{-4t}, e^{2t}\right\} = \left\{te^{2t}\right\}.$$

Thus, a particular solution of the nonhomogeneous equation will have the form $y_p(t) = Ate^{2t}$, and the unknown constant A can be determined by substituting this back in the differential equation. Compute the derivatives of $y_p(t)$:

$$y_p(t) = Ate^{2t}, \qquad y_p'(t) = A(e^{2t} + 2te^{2t}), \qquad y_p''(t) = A(4e^{2t} + 4te^{2t}).$$

Substituting in the differential equation gives

$$\begin{aligned} 4e^{2t} &= y_p'' - y_p' - 2y_p \\ &= A(4e^{2t} + 4te^{2t}) - A(e^{2t} + 2te^{2t}) - 2Ate^{2t} \\ &= 3Ae^{2t}. \end{aligned}$$

Solving for A gives $A = 4/3$. Thus, $y_p(t) = (4/3)te^{2t}$ and the general solution is

$$\boxed{y(t) = y_h(t) + y_p(t) = c_1e^{-t} + c_2e^{2t} + \frac{4}{3}te^{2t}.}$$

◀

7. [**12 Points**] Find a 2π-periodic solution of the differential equation

$$y'' + 3y = \sum_{n=1}^{\infty} \frac{1}{n} \sin nt.$$

▶ **Solution.** A 2π-periodic solution y_p can be expressed as a Fourier series

$$y_p = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt).$$

Compute the second derivative of y_p by term-by-term differentiation to get

$$y_p'' = \sum_{n=1}^{\infty}(-n^2 a_n \cos nt - n^2 b_n \sin nt).$$

Substituting these into the differential equation gives

$$\begin{aligned} y_p'' + 3y_p &= \frac{3a_0}{2} + \sum_{n=1}^{infty}((3-n^2)a_n \cos nt + (3-n^2)b_n \sin nt) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sin nt. \end{aligned}$$

Since there is no constant term on the right hand side, we have $3a_0/2 = 0$ so $a_0 = 0$. Since there are no cosine terms on the right, it follows that $(3-n^2)a_n = 0$ for all $n \geq 1$. Hence $a_n = 0$ for all $n \geq 0$. Equating the sine terms on the left and the right gives $(3-n^2)b_n = 1/n$ which gives

$$b_n = \frac{1}{n(3-n^2)}.$$

Thus, the period solution y_p is given by the Fourier series

$$\boxed{y_p(t) = \sum_{n=1}^{\infty} \frac{1}{n(3-n^2)} \sin nt.}$$

◀

8. [**12 Points**] Find the general solution of the differential equation

$$ty'' - y' = 3t^2 - 1,$$

given the fact that two solutions of the associated homogeneous equation are $y_1(t) = 1$ and $y_2(t) = t^2$.

▶ **Solution.** Use variation of parameters. First divide by t to put the equation in standard form

$$y'' - \frac{1}{t}y' = 3t = \frac{1}{t}.$$

A particular solution is given by

$$y_p = u_1 + u_2 t^2$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1' + u_2' t^2 &= 0 \\ 2tu_2' &= 3t - \frac{1}{t}. \end{aligned}$$

Thus,

$$u_2' = \frac{3}{2} - \frac{1}{2t^2}$$

and then

$$u_1' = -t^2 u_2' = -\frac{3}{2}t^2 + \frac{1}{2}.$$

Integrating gives $u_1 = -\frac{1}{2}t^3 + \frac{1}{2}t$ and $u_2' = \frac{3}{2}t + 1/2t$ and

$$y_p = u_1 + u_2 t^2 = -\frac{1}{2}t^3 + \frac{1}{2}t + \frac{3}{2}t^3 + \frac{1}{2}t = t^3 + t.$$

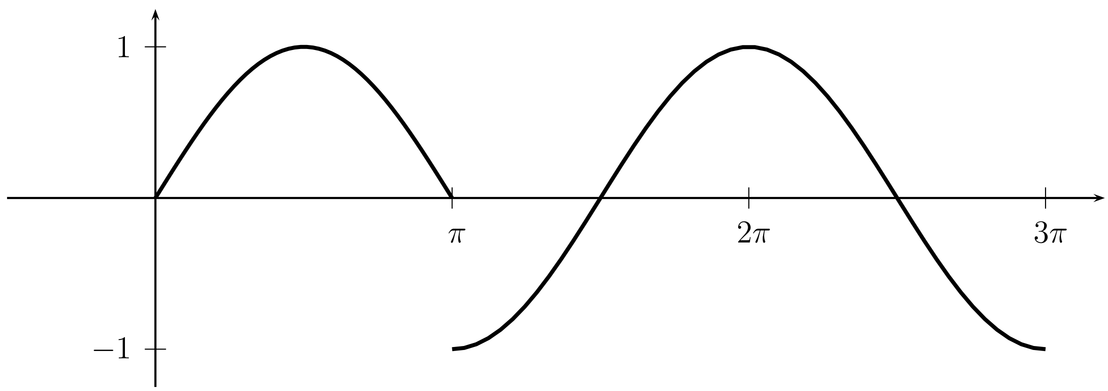
◀

9. [**12 Points**] Let $f(t)$ be the following function:

$$f(t) = \begin{cases} \sin t & \text{if } 0 \le t < \pi, \\ \cos t & \text{if } t \ge \pi. \end{cases}$$

(a) Sketch the graph of $f(t)$ on the interval $[0,\, 3\pi]$.

▶ **Solution.** Graph of $f(t)$:



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(b) Compute the Laplace transform of $f(t)$.

▶ **Solution.** Write $f(t)$ using the Heaviside function:

$$\begin{aligned} f(t) &= (\sin t)\chi_{[0,\,\pi)}(t) + (\cos t)\chi_{[0,\,\infty)} \\ &= (\sin t)(h(t) - h(t-\pi)) + (\cos t)h(t-\pi) \\ &= \sin t - \sin t h(t-\pi) + \cos t h(t-\pi) \end{aligned}$$

Then,

$$\begin{aligned} F(s) = \mathcal{L}\left\{f(t)\right\} &= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\left\{\sin(t+\pi)\right\} + e^{-\pi s}\mathcal{L}\left\{\cos(t+\pi)\right\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\left\{-\sin t\right\} + e^{-\pi s}\mathcal{L}\left\{-\cos t\right\} \\ &= \boxed{\frac{1}{s^2+1} + \frac{1}{s^2+1}e^{-\pi s} - \frac{s}{s^2+1}e^{-\pi s}.} \end{aligned}$$

◀

10. [**10 Points**] Compute the following inverse Laplace transform:

$$\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}e^{-3s}\right\}$$

▶ **Solution.** Let $F(s) = \dfrac{3s+1}{s^2+4}$. Then

$$F(s) = \frac{3s}{s^2+4} + \frac{1}{s^2+4}$$

so

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = 3\cos 2t + \frac{1}{2}\sin 2t,$$

and hence

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{3s+1}{s^2+4}e^{-3s}\right\} &= \mathcal{L}^{-1}\left\{F(s)e^{-3s}\right\} \\ &= f(t-3)h(t-3) \\ &= \left(3\cos 2(t-3) + \frac{1}{2}\sin 2(t-3)\right)h(t-3).\end{aligned}$$

◀

11. [**12 Points**] Let $A = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}$.

(a) Compute e^{At}.

▶ **Solution.** Use Fulmer's method. $sI - A = \begin{bmatrix} s-2 & 5 \\ -4 & s+2 \end{bmatrix}$ so $q(s) = \det(sI - A) = (s-2)(s+2) + 20 = s^2 + 16$. Then $\mathcal{B}_q = \{\cos 4t, \sin 4t\}$ so $e^{At} = M_1 \cos 4t + M_2 \sin 4t$. Differentiating gives $Ae^{At} = -4M_1 \sin 4t + 4M_2 \cos 4t$, and evaluating both of these at $t = 0$ gives the equations

$$\begin{aligned} I &= M_1 \\ A &= 4M_2. \end{aligned}$$

Thus, $M_1 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $M_2 = \frac{1}{4}A = \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ 1 & -\frac{1}{2} \end{bmatrix}$. Hence

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\cos 4t + \begin{bmatrix} \frac{1}{2} & -\frac{5}{4} \\ 1 & -\frac{1}{2} \end{bmatrix}\sin 4t \\ &= \begin{bmatrix} \cos 4t + \frac{1}{2}\sin 4t & -\frac{5}{4}\sin 4t \\ \sin 4t & \cos 4t - \frac{1}{2}\sin 4t \end{bmatrix}. \end{aligned}$$

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(b) Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

▶ **Solution.**

$$
\begin{aligned}
\mathbf{y}(t) &= e^{At}\mathbf{y}(0) \\
&= \begin{bmatrix} \cos 4t + \frac{1}{2}\sin 4t & -\frac{5}{4}\sin 4t \\ \sin 4t & \cos 4t - \frac{1}{2}\sin 4t \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} \cos 4t + 3\sin 4t \\ -2\cos 4t + 2\sin 4t \end{bmatrix}.
\end{aligned}
$$

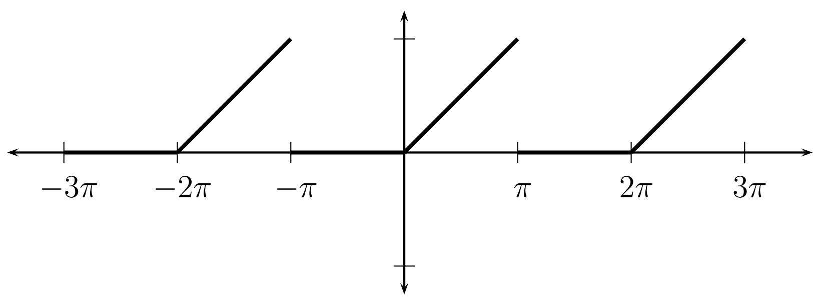
◀

12. [**12 Points**]Let $f(t)$ be the periodic function of period 2π that is defined on the interval $(-\pi,\, \pi]$ by

$$
f(t) = \begin{cases} 0 & \text{if } -\pi < t \le 0, \\ t & \text{if } 0 < t \le \pi. \end{cases}
$$

(a) Sketch the graph of $f(t)$ on the interval $[-3\pi,\, 3\pi]$.

▶ **Solution.** Graph of $f(t)$:



◀

(b) Compute the Fourier series of $f(t)$. The following integration formulas may be of use:

$$
\int t\sin at\,dt = \frac{1}{a^2}\sin at - \frac{t}{a}\cos at \qquad \int t\cos at\,dt = \frac{1}{a^2}\cos at + \frac{t}{a}\sin at.
$$

▶ **Solution.** Since $f(t)$ is periodic of period 2π, the Fourier series has the form

$$
f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)
$$

where

$$
a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad \text{and} \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt
$$

for all $n \ge 1$. For the given $f(t)$, the function $f(t)$ is 0 on the interval $-\pi < t \ge 0$, so the integrals can be replaced by an integral from 0 to π. Thus,

$$
a_0 = \frac{1}{\pi}\int_0^{\pi} t\,dt = \left.\frac{t^2}{2\pi}\right|_0^{\pi} = \frac{\pi}{2},
$$

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt \\
&= \frac{1}{\pi}\left[\frac{1}{n^2}\cos nt + \frac{t}{n}\sin nt\right]_0^{\pi} \\
&= \frac{\cos n\pi - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2},
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt \\
&= \frac{1}{\pi}\left[\frac{1}{n^2}\sin nt - \frac{t}{n}\cos nt\right]_0^{\pi} \\
&= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}.
\end{aligned}$$

Thus, the Fourier series of $f(t)$ is

$$f(t) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nt + \frac{(-1)^{n+1}}{n}\sin nt\right).$$

◀

(c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(0)$, $g(\pi)$, and $g(5\pi/2)$.

▶ **Solution.** Since $f(t)$ is continuous at $t = 0$, the series sums to $f(0)$. Thus, $g(0) = f(0) = 0$. $f(t)$ has a jump discontinuity at $t = \pi$. Thus

$$g(\pi) = \frac{f(\pi^+) + f(\pi^-)}{2} = \frac{\pi}{2}.$$

$f(t)$ is continuous at $t = 5\pi/2$ so

$$g\left(\frac{5\pi}{2}\right) = f\left(\frac{5\pi}{2}\right) = f\left(\frac{5\pi}{2} - 2\pi\right) = \left(\frac{\pi}{2}\right) = \frac{\pi}{2}.$$

◀

13. [**12 Points**] A 10-liter jug of water is contaminated with 100 grams of salt. Assume that the jug is always well-mixed. Starting at time $t = 0$, pure water begins flowing into the jug at a rate of 2 liters per minute. Well-mixed salty water flows out of the jug at the rate of 3 liters per minute. This means that the entire jug will be empty at time $t = 10$ minutes. How much salt will be in the jug at time $t = 5$ minutes?

▶ **Solution.** Let $y(t)$ be the amount of salt in the tank at time t, measured in grams. Our goal is to find $y(5)$. First determine the differential equation that governs $y(t)$. The balance equation is

$$y'(t) = \text{rate in} - \text{rate out}.$$

The rate in is 0 since pure water is being added. The rate out is

$$\left(\frac{y(t)}{V(t)}\right) \times 3.$$

Since mixture is entering at a rate of 2 liters per minute and leaving at the volume rate of 3 liters per minute, the volume of mixture in the tank is $10 - t$ liters at time t. Thus $V(t) = 10 - t$. Hence $y(t)$ satisfies the equation

$$y' = -\frac{3}{10 - t} y,$$

so that the initial value problem satisfied by $y(t)$ is

$$y' + \frac{3}{10 - t} y = 0, \qquad y(0) = 100.$$

This is a linear differential equation with $p(t) = 3/(10 - t)$ so that $P(t) = \int p(t) = -3 \ln |10 - t| = \ln |10 - t|^{-3}$. Therefore the integrating factor is $\mu(t) = e^{\ln |10 - t|^{-3}} = |10 - t|^{-3}$, so multiplication of the differential equation by $\mu(t)$ gives an equation

$$\frac{d}{dt}\left(|10 - t|^{-3} y\right) = 0.$$

Integration of this equation gives

$$|10 - t|^{-3} y = C,$$

where C is an integration constant. Thus

$$y(t) = |10 - t|^3 C$$

and the initial condition $y(0) = 100$ gives

$$100 = y(0) = C \cdot 10^3$$

so that $C = 1/10$, and

$$y(t) = \frac{1}{10}(10 - t)^3.$$

Then the amount of salt at time $t = 5$ is

$$y(5) = \frac{1}{10} \cdot 5^3 = 12.5 \text{ grams.}$$

◀

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.