Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper.

1. [**15 Points**] Each of the following matrices is in reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -4 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 5 & 0 & -3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

For each of these matrices, write down all solutions to the linear system of equations that has the given matrix as augmented matrix.

▶ **Solution.** For A: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix}$.

For B: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4\alpha + 2 \\ \alpha \\ -5 \end{bmatrix}$ where $x_2 = \alpha$ is arbitrary.

For C: The last row corresponds to the equation $0 = 1$. Thus the system has no solutions. ◀

2. [**15 Points**] Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}$.

(a) Compute A^{-1}.

▶ **Solution.**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 3 & 8 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & -1 & -3 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & 7 & -5 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 0 & 8 & -5 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 10 & -6 & 1 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 5 & -1 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}.$$

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(b) Using your answer to part (a), solve the linear system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

▶ **Solution.** $\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 37 \\ -7 \\ -27 \end{bmatrix}$. ◀

3. [**10 Points**] Let $C = \begin{bmatrix} 0 & 0 & 5 & 2 \\ 2 & 4 & 6 & 2 \\ 1 & -1 & 6 & 0 \\ 0 & 0 & -3 & 2 \end{bmatrix}$. Compute $\det C$.

▶ **Solution.** Use cofactor expansion along the first row to get

$$\begin{aligned} \det C &= 5 \det \begin{bmatrix} 2 & 4 & 2 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} - (2) \det \begin{bmatrix} 2 & 4 & 6 \\ 1 & -1 & 6 \\ 0 & 0 & -3 \end{bmatrix} \\ &= 2 \cdot 5 \det \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} - (2)(-3) \det \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} = 16 \det \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix} \\ &= 16(-2-4) = 16(-6) = -96. \end{aligned}$$

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4. [**20 Points**] Solve the initial value problem: $y' + 3y = 12e^{3t} - 3e^{-3t}, \quad y(0) = -3$.

▶ **Solution.** This equation is linear with coefficient function $p(t) = 3$ so that an integrating factor is given by $\mu(t) = e^{\int 3\,dt} = e^{3t}$. Multiplication of the differential equation by the integrating factor gives

$$e^{3t}y' + 3e^{3t}y = 12e^{6t} - 3,$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(e^{3t}y) = 12e^{6t} - 3.$$

Integration then gives

$$e^{3t}y = 2e^{6t} - 3t + C,$$

where C is an integration constant. Multiplying through by e^{-3t} gives

$$y = 2e^{3t} - 3t + Ce^{-3t}.$$

The initial condition gives $-3 = y(0) = 2 + C$ so $C = -5$ and thus

$$\boxed{y = 2e^{3t} - 3te^{-3t} - 5e^{-3t}.}$$

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5. [**20 Points**] Solve the initial value problem: $ty' + 6y = 11t^5 \quad y(1) = 3$.

► **Solution.** This equation is linear. Putting it in standard form gives

$$y' + \frac{6}{t}y = 11t^4.$$

Thus, the coefficient function is $p(t) = \frac{6}{t}$, so that an integrating factor is $\mu(t) = e^{\int \frac{6}{t}\,dt} = e^{6\ln|t|} = e^{\ln(|t|^6)} = |t|^6 = t^6$. Multiplication of the differential equation (in standard form) by the integrating factor gives

$$t^6y' + 6t^5y = 11t^{10},$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(t^6y) = 11t^{10}.$$

Integration then gives

$$t^6y = t^{11} + C,$$

where C is an integration constant. Multiplying through by t^{-6} gives

$$y = t^5 + Ct^{-6}.$$

The initial condition $y(1) = 3$ implies $3 = y(1) = 1 + C$ so that $C = 2$. Hence

$$\boxed{y = t^5 + 2t^{-6}.}$$

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6. [**20 Points**] Solve the initial value problem: $(1 + t^2)y' = -2ty^2, \quad y(0) = 1$.

► **Solution.** This equation is separable. There is one equilibrium solution: $y = 0$. This does not satisfy the initial condition so it is not the required solution. If $y \neq 0$, separate variables to get $y^{-2}y' = \frac{-2t}{1+t^2}$, which in differential form is

$$y^{-2}\,dy = \frac{-2t}{1+t^2}\,dt$$

and integration then gives

$$-y^{-1} = -\int \frac{2t}{1+t^2}\,dt = -\ln(1+t^2) + C,$$

so that

$$y = \frac{1}{\ln(1+t^2) - C}.$$

The initial condition $y(0) = 1$ gives $1 = 1/(-C)$ so $C = -1$, and

$$\boxed{y = \frac{1}{\ln(1+t^2) + 1}.}$$

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