Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and a convolution table have been appended to the exam.

1. Let $A = \begin{bmatrix} 6 & 5 \\ -8 & -6 \end{bmatrix}$.

   (a) **[3 Points]** Verify that the characteristic polynomial of A is $c_A(s) = s^2 + 4$.

   ▶ **Solution.**

   $$c_A(s) = \det(sI - A) = \det \begin{bmatrix} s-6 & -5 \\ 8 & s+6 \end{bmatrix} = (s-6)(s+6) + 40 = s^2 + 4.$$

   ◀

   (b) **[10 Points]** Compute the matrix exponential e^{At}.

   ▶ **Solution.** Since $c_A(s) = s^2 + 4$,

   $$(sI - A)^{-1} = \frac{1}{s^2+4} \begin{bmatrix} s+6 & 5 \\ -8 & s-6 \end{bmatrix} = \begin{bmatrix} \frac{s+6}{s^2+4} & \frac{5}{s^2+4} \\ \frac{-8}{s^2+4} & \frac{s-6}{s^2+4} \end{bmatrix}.$$

   Then

   $$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} \cos 2t + 3\sin 2t & \frac{5}{2}\sin 2t \\ -4\sin 2t & \cos 2t - 3\sin 2t \end{bmatrix}.$$

   ◀

   (c) **[7 Points]** Solve the initial value problem $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

   ▶ **Solution.**

   $$\mathbf{y} = e^{At}\mathbf{y}(0) = \begin{bmatrix} \cos 2t + 3\sin 2t & \frac{5}{2}\sin 2t \\ -4\sin 2t & \cos 2t - 3\sin 2t \end{bmatrix} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3\cos 2t + 4\sin 2t \\ -2\cos 2t - 6\sin 2t \end{bmatrix}.$$

   ◀

   (d) **[10 Points]** Solve the initial value problem $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t)$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $\mathbf{f}(t) = \begin{bmatrix} \sin 2t \\ 0 \end{bmatrix}$.

▶ **Solution.**

$$
\begin{aligned}
\mathbf{y}(t) &= e^{At}\mathbf{y}(0) + e^{At} * \mathbf{f}(t) \\
&= \begin{bmatrix} \cos 2t + 3\sin 2t & \frac{5}{2}\sin 2t \\ -4\sin 2t & \cos 2t - 3\sin 2t \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \cos 2t + 3\sin 2t & \frac{5}{2}\sin 2t \\ -4\sin 2t & \cos 2t - 3\sin 2t \end{bmatrix} * \begin{bmatrix} \sin 2t \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} (\cos 2t) * \sin 2t + 3(\sin 2t) * \sin 2t \\ -4(\sin 2t) * \sin 2t \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}t\sin 2t + \frac{3}{4}(\sin 2t - 2t\cos 2t) \\ -(\sin t - 2t\cos 2t) \end{bmatrix},
\end{aligned}
$$

where the convolutions are computed from numbers 14 and 16 in the Table of Convolutions. ◀

2. Consider the following first order linear system of differential equations

$$
\begin{aligned}
y_1' &= 7y_1 + y_2 \\
y_2' &= -4y_1 + 3y_2
\end{aligned}
$$

(a) [**5 Points**] Write the system in matrix form $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.**

$$
\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.
$$

◀

(b) [**20 Points**] Solve this system with the initial conditions $y_1(0) = c_1$, $y_2(0) = c_2$.

▶ **Solution.** First compute e^{At}, where $A = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}$. We will use Fulmer's method. The characteristic polynomial of A is

$$
c_A(s) = \det(sI - A) = \det \begin{bmatrix} s-7 & 1 \\ -4 & s-3 \end{bmatrix} = (s-7)(s-3) + 4 = s^2 - 10s + 25 = (s-5)^2.
$$

Thus, the only root is 5 with multiplicity 2, so that $\mathcal{B}_{c_A(s)} = \{e^{5t}, te^{5t}\}$. Thus

$$
e^{At} = M_1 e^{5t} + M_2 te^{5t},
$$

and differentiating gives

$$
Ae^{At} = 5M_1 e^{5t} + M_2(e^{5t} + 5te^{5t}).
$$

Evaluating at $t = 0$ gives the equations for M_1 and M_2:

$$
\begin{aligned}
I &= M_1 \\
A &= 5M_1 + M_2.
\end{aligned}
$$

Thus, $M_1 = I$ and $M_2 = A - 5I$ so that

$$\begin{aligned} e^{At} &= Ie^{5t} + (A - 5I)te^{5t} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{5t} + \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix} te^{5t} \\ &= \begin{bmatrix} e^{5t} + 2te^{5t} & te^{5t} \\ -4te^{5t} & e^{5t} - 2te^{5t} \end{bmatrix}. \end{aligned}$$

Then the solution of the system is

$$\begin{aligned} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} &= e^{At} \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} \\ &= \begin{bmatrix} e^{5t} + 2te^{5t} & te^{5t} \\ -4te^{5t} & e^{5t} - 2te^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_1 \end{bmatrix} \\ &= \begin{bmatrix} c_1e^{5t} + (2c_1 + c_2)te^{5t} \\ c_2e^{5t} + (-4c_1 - 2c_2)te^{5t} \end{bmatrix}. \end{aligned}$$

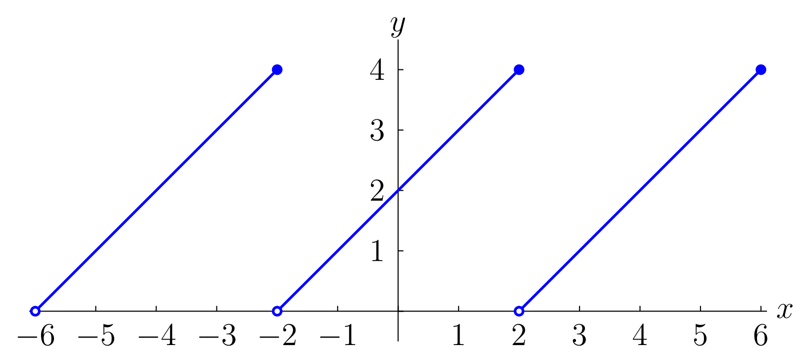
◀

3. Let $f(x)$ be the periodic function of period 4 that is defined on the interval $(-2, 2]$ by

$$f(x) = 2 + x.$$

(a) [**5 Points**] Sketch the graph of $f(x)$ on the interval $[-6, 6]$.

▶ **Solution.** ◀



(b) [**14 Points**] Compute the Fourier series of $f(x)$. The following integration formulas may be of use:

$$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax \qquad \int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax.$$

▶ **Solution.** The period of $f(x)$ is 4, so $L = 2$. Then

$$a_0 = \frac{1}{2} \int_{-2}^{2} f(x) \, dx = \frac{1}{2} \int_{-2}^{2} (2 + x) \, dx = \frac{1}{2} \left(2x + \frac{x^2}{2} \right) \Bigg|_{-2}^{2} = 4.$$

For $n \geq 1$,

$$\begin{aligned} a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{-2}^{2} (2+x)\cos\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{-2}^{2} 2\cos\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{-2}^{2} x\cos\frac{n\pi}{2}x\,dx \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{-2}^{2} (2+x)\sin\frac{n\pi}{2}x\,dx \\ &= \frac{1}{2}\int_{-2}^{2} 2\sin\frac{n\pi}{2}x\,dx + \frac{1}{2}\int_{-2}^{2} x\sin\frac{n\pi}{2}x\,dx \\ &= 0 + \int_{0}^{2} x\sin\frac{n\pi}{2}x\,dx \\ &= \left[\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{2}{n\pi}x\cos\frac{n\pi}{2}x\right]\Bigg|_0^2 \\ &= \frac{-4}{n\pi}\cos n\pi = \frac{4(-1)^{n+1}}{n\pi}. \end{aligned}$$

Thus, the Fourier series of $f(x)$ is

$$f(x) \sim 2 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}x.$$

◀

(c) **[6 Points]** Let $g(x)$ denote the sum of the Fourier series found in part (b). Compute $g(0)$, $g(2)$, and $g(5)$.

▶ **Solution.** f is continuous at $x = 0$ so the Fourier series converges to $f(x)$ at $x = 0$. Thus, $g(0) = f(0) = 2$. For $x = 2$,

$$g(2) = \frac{f(2^+) + f(2^-)}{2} = \frac{0+4}{2} = 2.$$

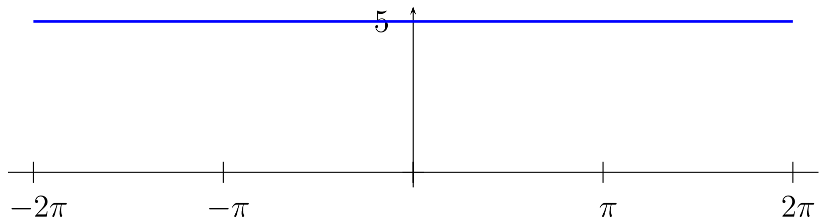
f is continuous at $x = 5$ so $g(5) = f(5) = f(1) = 3$. ◀

4. Consider the function

$$f(x) = 5, \qquad 0 < x < \pi.$$

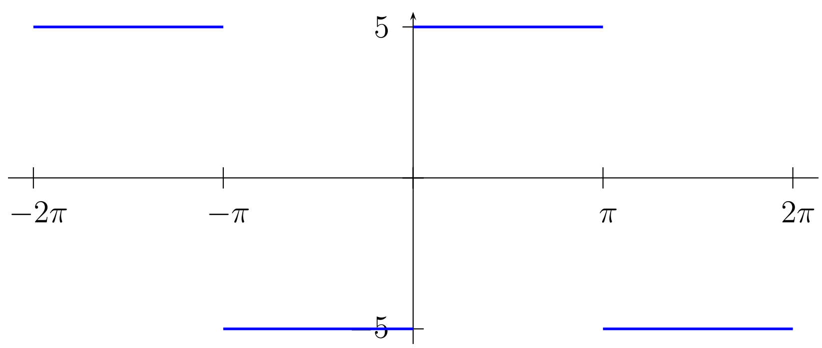
(a) [**4 Points**] Let $g(x)$ be the even periodic extension of $f(x)$. Draw the graph of $g(x)$ on the interval $[-2\pi, 2\pi]$.

▶ **Solution.** ◀



(b) [**4 Points**] Let $h(x)$ be the odd periodic extension of $f(x)$. Draw the graph of $h(x)$ on the interval $[-2\pi, 2\pi]$.

▶ **Solution.** ◀



(c) [**12 Points**] Compute the Fourier sine series of $f(x)$.

▶ **Solution.** $L = \pi$ so

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} 5\sin nx\,dx \\ &= \frac{10}{\pi}\left[\frac{-1}{n}\cos nx\right]\Big|_0^{\pi} \\ &= \frac{10}{n\pi}(-\cos n\pi + 1) \\ &= \frac{10}{n\pi}(1-(-1)^n) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-20}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier Sine series of $f(x)$ is

$$f(x) \sim \frac{-20}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nx.$$

◀

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}.$
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s).$

Table of Convolutions

	$f(t)$	$g(t)$	$(f * g)(t)$
1.	1	$g(t)$	$\int_0^t g(\tau)\,d\tau$
2.	t^m	t^n	$\frac{m!n!}{(m+n+1)!}t^{m+n+1}$
3.	t	$\sin at$	$\frac{at - \sin at}{a^2}$
4.	t^2	$\sin at$	$\frac{2}{a^3}(\cos at - (1 - \frac{a^2t^2}{2}))$
5.	t	$\cos at$	$\frac{1 - \cos at}{a^2}$
6.	t^2	$\cos at$	$\frac{2}{a^3}(at - \sin at)$
7.	t	e^{at}	$\frac{e^{at} - (1 + at)}{a^2}$
8.	t^2	e^{at}	$\frac{2}{a^3}(e^{at} - (a + at + \frac{a^2t^2}{2}))$
9.	e^{at}	e^{bt}	$\frac{1}{b-a}(e^{bt} - e^{at}) \quad a \neq b$
10.	e^{at}	e^{at}	te^{at}
11.	e^{at}	$\sin bt$	$\frac{1}{a^2+b^2}(be^{at} - b\cos bt - a\sin bt)$
12.	e^{at}	$\cos bt$	$\frac{1}{a^2+b^2}(ae^{at} - a\cos bt + b\sin bt)$
13.	$\sin at$	$\sin bt$	$\frac{1}{b^2-a^2}(b\sin at - a\sin bt) \quad a \neq b$
14.	$\sin at$	$\sin at$	$\frac{1}{2a}(\sin at - at\cos at)$
15.	$\sin at$	$\cos bt$	$\frac{1}{b^2-a^2}(a\cos at - a\cos bt) \quad a \neq b$
16.	$\sin at$	$\cos at$	$\frac{1}{2}t\sin at$
17.	$\cos at$	$\cos bt$	$\frac{1}{a^2-b^2}(a\sin at - b\sin bt) \quad a \neq b$
18.	$\cos at$	$\cos at$	$\frac{1}{2a}(at\cos at + \sin at)$