Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $ty' - 2y = 2t^3, \quad y(1) = 1.$
2. [**12 Points**] $(1+t)y' - y^2 = 0, \quad y(0) = -1/2.$
3. [**12 Points**] $y'' + 2y' + 6y = 0, \quad y(0) = 0,\ y'(0) = 2.$
4. [**10 Points**] $t^2y'' - 2y = 0.$
5. [**12 Points**] $y'' + 4y = \delta(t-4), \qquad y(0) = 0,\ y'(0) = 1.$
6. [**12 Points**] $y'' + y' - 2y = e^t + 2t.$
7. [**12 Points**] Find a particular solution for $t > 0$ of the differential equation

$$y'' - \left(1 + \frac{2}{t}\right)y' + \left(\frac{1}{t} + \frac{2}{t^2}\right)y = 2t,$$

given the fact that two solutions of the associated homogeneous equation are $y_1(t) = t$ and $y_2(t) = te^t$.

8. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:

$$f(t) = \begin{cases} 2t^2 & \text{if } 0 \le t < 3, \\ 4 & \text{if } t \ge 3. \end{cases}$$

9. [**12 Points**] Compute the inverse Laplace transform of the following functions:
   (a) $F(s) = \dfrac{s}{(s^2 + 4s + 5)}.$
   (b) $G(s) = \dfrac{s+3}{(s^2+1)s}.$
10. [**12 Points**] Solve the following system of differential equations

$$\begin{aligned} y_1' &= -2y_1 - 4y_2 \qquad & y_1(0) &= 0, \\ y_2' &= 5y_1 + 7y_2 \qquad & y_2(0) &= 2. \end{aligned}$$

11. [**12 Points**]Let $f(t)$ be the periodic function of period 4π that is defined on the interval $(-2\pi, 2\pi]$ by

$$f(t) = \begin{cases} 2 & \text{if } -2\pi < t \le 0, \\ 0 & \text{if } 0 < t \le 2. \end{cases}$$

   (a) Sketch the graph of $f(t)$ on the interval $[-4\pi, 4\pi]$.

   (b) Compute the Fourier series of $f(t)$.

   (c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(2\pi)$ and $g(3\pi)$.

12. [**12 Points**] A 400-gallon tank initially contains 100 gallons of brine with a concentration of 1.5 ounces of salt per gallon. Starting at time $t = 0$, brine with a salt concentration of 2.0 ounces of salt per gallon runs into the tank at the rate of 6 gallons per minute. The well-mixed solution is drawn off at the same rate of 6 gallons per minute. Let $y(t)$ denote the number of ounces of salt in the tank at time t.

   (a) What is $y(0)$?

   (b) What is the differential equation that $y(t)$ satisfies?

   (c) Solve this differential equation to determine the amount of salt in the tank at time t.

   (d) Find the amount of salt in the tank after 20 minutes.

Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\frac{1}{s}$
2.	t^n	$\to$	$\frac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\frac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\frac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\frac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\frac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\frac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\frac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\frac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\frac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$