Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**12 Points**] $ty' - 2y = 2t^3, \quad y(1) = 1.$

▶ **Solution.** This equation is linear. First put it in standard form by dividing by t to get

$$y' - \frac{2}{t}y = 2t^2.$$

To find an integrating factor note that $p(t) = -2/t$ so that an integrating factor is

$$\mu(t) = e^{\int p(t)\,dt} = e^{-\int 2/t\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = t^{-2}.$$

Multiplying the equation by t^{-2} gives

$$t^{-2}y' - \frac{2}{t^3}y = 2.$$

Thus,

$$\left(t^{-2}y\right)' = 2,$$

and integrating gives

$$t^{-2}y = 2t + C.$$

Solving for y gives

$$y = 2t^3 + ct^2.$$

Now apply the initial condition $y(1) = 1$ to get $1 = y(1) = 2 + C$. Therefore, $C = -1$ and

$$\boxed{y(t) = 2t^3 - t^2.}$$

◀

2. [**12 Points**] $(1+t)y' - y^2 = 0, \quad y(0) = -1/2.$

▶ **Solution.** This equation is separable, which is seen by rewriting the equation as

$$y' = \frac{1}{1+t}y^2.$$

Separating the variables gives

$$y^{-2}y' = \frac{1}{1+t},$$

which, in differential form is $y^{-2}\,dy = \dfrac{dt}{1+t}$. Integration gives

$$\int y^{-2}\,dy = \int \frac{1}{1+t}\,dt \implies -\frac{1}{y} = \ln(1+t) + C.$$

The initial condition $y(0) = -1/2$ implies that

$$2 = \ln(1) + C \implies C = 2.$$

Solving for y then gives

$$\boxed{y(t) = \frac{-1}{\ln(1+t)+2}.}$$

◀

3. [**12 Points**] $y'' + 2y' + 6y = 0, \quad y(0) = 0,\ y'(0) = 2.$

▶ **Solution.** The characteristic polynomial is $q(s) = s^2 + 2s + 6 = (s+1)^2 + 5$, which has roots $-1 \pm \sqrt{5}i$. Thus, the general solution is given by

$$y = c_1 e^{-t}\cos\sqrt{5}t + c_2 e^{-t}\sqrt{5}t.$$

The initial condition $y(0) = 0$ gives $c_1 = 0$ so that $y = c_2 e^{-t}\sin\sqrt{5}t$. Then

$$y' = c_2(-e^{-t}\sin\sqrt{5}t + \sqrt{5}e^{-t}\cos\sqrt{5}t),$$

so the initial condition $y'(0) = 2$ gives the equation $2 = y'(0) = \sqrt{5}c_2$, which implies that $c_2 = 2/\sqrt{5}$. Hence

$$\boxed{y(t) = \frac{2}{\sqrt{5}}e^{-t}\sin\sqrt{5}t.}$$

◀

4. [**10 Points**] $t^2y'' - 2y = 0.$

▶ **Solution.** This is a Cauchy-Euler equation with indicial equation $Q(s) = s(s-1) - 2 = s^2 - s - 2 = (s-2)(s+1)$ which has roots 2 and -1. The general solution is thus

$$\boxed{c_1t^2 + c_2t^{-1}.}$$

◀

5. [**12 Points**] $y'' + 4y = \delta(t-4), \qquad y(0) = 0,\ y'(0) = 1.$

▶ **Solution.** Use the Laplace transform method. Applying the Laplace transform to both sides of the differential equation gives (letting $Y(s) = \mathcal{L}\{y(t)\}$)

$$\begin{aligned} s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= e^{-4s} && \Longrightarrow \\ (s^2+4)Y(s) &= 1 + e^{-4s} && \Longrightarrow \\ Y(s) &= \frac{1}{s^2+4} + \frac{1}{s^2+4}e^{-4s}. \end{aligned}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin 2t$, the inverse form of the second translation theorem gives

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}e^{-4s}\right\} = h(t-4)\cdot\frac{1}{2}\sin 2(t-4).$$

Thus,

$$\begin{aligned} y(t) &= \frac{1}{2}\sin 2t + h(t-4)\cdot\frac{1}{2}\sin 2(t-4) \\ &= \begin{cases} \frac{1}{2}\sin 2t & \text{if } 0 \le t < 4, \\ \frac{1}{2}\sin 2t = \frac{1}{2}\sin 2(t-4) & \text{if } t \ge 4. \end{cases} \end{aligned}$$

◀

6. [**12 Points**] $y'' + y' - 2y = e^t + 2t$.

▶ **Solution.** Use undetermined coefficients. The homogeneous equation has characteristic equation $q(s) = s^2+s-2 = (s+2)(s-1)$, which has roots -2 and 1. Thus, the standard basis $\mathcal{B}_q = \{e^{-2t},\, e^t\}$ and $y_h = c_1e^{-2t} + c_2e^t$. Taking the Laplace transform of the right hand side gives

$$\mathcal{L}\left\{e^t + 2t\right\} = \frac{1}{s-1} + \frac{2}{s^2} = \frac{s^2+2s+2}{s^2(s-1)},$$

which has denominator $v(s) = s^2(s-1)$, so that $q(s)v(s) = s^2(s-1)^2(s+2)$. Hence,

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{1,\, t,\, e^t,\, te^t,\, e^{-2t}\right\} \setminus \left\{e^t,\, e^{-2t}\right\} = \left\{1,\, t,\, te^t\right\}.$$

Thus, a particular solution $y_p(t)$ will have the form $y_p = A + Bt + Cte^t$ for some constants A, B, C to be determined. To find the constants, compute

$$\begin{aligned} y_p' &= B + C(e^t + te^t) \\ y_p'' &= C(2e^t + te^t). \end{aligned}$$

Substitute into the original equation to get

$$\begin{aligned} y_p'' + y_p' - 2y_p &= c(2e^t + te^t) + B + C(e^t + te^t) - 2(A + Bt + Cte^t) \\ &= 3Ce^t + (B - 2A) - 2Bt \\ &= e^t + 2t. \end{aligned}$$

Equating coefficients of like terms gives $3C = 1$, $B - 2A = 0$, $-2B = 2$. Thus $C = 1/3$, $B = -1$, and $A = \frac{1}{2}B = -\frac{1}{2}$. Thus, the particular solution is

$$y_p = \frac{1}{3}te^t - t - \frac{1}{2}.$$

Since $y_g = y_h + y_p$, the general solution is

$$\boxed{y_g(t) = c_1e^t + c_2e^{-2t} + \frac{1}{3}te^t - t - \frac{1}{2}.}$$

◀

7. [**12 Points**] Find a particular solution for $t > 0$ of the differential equation

$$y'' - \left(1 + \frac{2}{t}\right)y' + \left(\frac{1}{t} + \frac{2}{t^2}\right)y = 2t,$$

given the fact that two solutions of the associated homogeneous equation are $y_1(t) = t$ and $y_2(t) = te^t$.

▶ **Solution.** Use variation of parameters. A particular solution is given by

$$y_p = u_1t + u_2te^t$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1't + u_2'te^t &= 0 \\ u_1' + u_2'(t+1)e^t &= 2t. \end{aligned}$$

Solve for u_1' and u_2' by means of Cramer's formula:

$$u_1' = \frac{\begin{vmatrix} 0 & te^t \\ 2t & (t+1)e^t \end{vmatrix}}{\begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix}} = \frac{-2t^2e^t}{t^2e^t} = -2$$

$$u_2' = \frac{\begin{vmatrix} t & 0 \\ 1 & 2t \end{vmatrix}}{\begin{vmatrix} t & te^t \\ 1 & (t+1)e^t \end{vmatrix}} = \frac{2t^2}{t^2e^t} = 2e^{-t}.$$

Integrating gives $u_1 = -2t$ and $u_2 = -2e^{-t}$. Thus,

$$\boxed{y_p(t) = -2t \cdot t + -2e^{-t} \cdot te^t = -2t^2 - 2t.}$$

◀

8. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:

$$f(t) = \begin{cases} 2t^2 & \text{if } 0 \le t < 3, \\ 4 & \text{if } t \ge 3. \end{cases}$$

▶ **Solution.** Write $f(t)$ in terms of the Heaviside step function:

$$\begin{aligned} f(t) &= 2t^2\chi_{[0,\,3)}(t) + 4\chi_{[3,\,\infty)}(t) \\ &= 2t^2(1 - h(t-3)) + 4h(t-3) \\ &= 2t^2 + (4 - 2t^2)h(t-3). \end{aligned}$$

Then the second translation theorem gives

$$\begin{aligned} F(s) &= \frac{4}{s^3} + e^{-3s}\mathcal{L}\left\{4 - 2(t+3)^2\right\} \\ &= \frac{4}{s^3} + e^{-3s}\mathcal{L}\left\{4 - 2(t^2 + 6t + 9)\right\} \\ &= \frac{4}{s^3} + e^{-3s}\left(\frac{4}{s} - \frac{4}{s^3} - \frac{12}{s^2} - \frac{18}{s}\right). \end{aligned}$$

Therefore,

$$\boxed{F(s) = \frac{4}{s^3} - e^{-3s}\left(\frac{14}{s} + \frac{12}{s^2} + \frac{4}{s^3}\right).}$$

◀

9. [**12 Points**] Compute the inverse Laplace transform of the following functions:

(a) $F(s) = \dfrac{s}{(s^2 + 4s + 5)}$.

▶ **Solution.** Complete the square to get $s^2 + 4s + 5 = (s+2)^2 + 1$. Then

$$\begin{aligned} F(s) &= \frac{s}{(s+2)^2 + 1} = \frac{(s+2) - 2}{(s+2)^2 + 1} \\ &= \frac{s+2}{(s+2)^2 + 1} - \frac{2}{(s+2)^2 + 1}, \end{aligned}$$

and

$$\boxed{f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = e^{-2t}\cos t - 2e^{-2t} sint.}$$

◀

(b) $G(s) = \dfrac{s+3}{(s^2+1)s}$.

▶ **Solution.** First expand $G(s)$ in a partial fraction expansion:

$$G(s) = \frac{A}{s} + \frac{p_1(s)}{s^2+1}$$

where

$$A = \left.\frac{s+3}{s^2+1}\right|_{s=0} = 3$$

and

$$p_1(s) = \frac{(s+3) - 3(s^2+1))}{s} = \frac{s-3s^2}{s} = 1-3s.$$

Thus

$$G(s) = \frac{3}{s} + \frac{1-3s}{s^2+1} = \frac{3}{s} + \frac{1}{s^2+1} - \frac{3s}{s^2+1}.$$

Then

$$\boxed{g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = 3 + \sin t - 3\cos t.}$$

◀

10. [**12 Points**] Solve the following system of differential equations

$$\begin{aligned} y_1' &= -2y_1 - 4y_2 \qquad & y_1(0) &= 0, \\ y_2' &= 5y_1 + 7y_2 \qquad & y_2(0) &= 2. \end{aligned}$$

▶ **Solution.** Let $A = \begin{bmatrix} -2 & -4 \\ 5 & 7 \end{bmatrix}$. Then the system becomes the matrix differential equation $\mathbf{y}' = A\mathbf{y}$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$. To solve this system, compute e^{At}. Use Fulmer's method. $sI - A = \begin{bmatrix} s+2 & 4 \\ -5 & s-7 \end{bmatrix}$ so $c_A(s) = \det(sI - A) = (s+2)(s-7) + 20 = s^2 - 5s + 6 = (s-2)(s-3)$. The roots are 2, 3. Then $\mathcal{B}_{c_A(s)} = \{e^{2t}, e^{3t}\}$ so $e^{At} = M_1e^{2t} + M_2e^{3t}$. Differentiating gives $Ae^{At} = 2M_1e^{2t} + 3M_2e^{3t}$, and evaluating both of these at $t = 0$ gives the equations

$$\begin{aligned} I &= M_1 + M_2 \\ A &= 2M_1 + 3M_2. \end{aligned}$$

Solving for M_1 and M_2 gives $M_2 = A - 2I$ and $M_1 = I - M_2 = I - (A - 2I) = 3I - A$. Thus, $M_2 = A - 2I = \begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix}$ and $M_1 = 3I - A = \begin{bmatrix} 5 & 4 \\ -5 & -4 \end{bmatrix}$. Hence

$$\begin{aligned} e^{At} &= \begin{bmatrix} 5 & 4 \\ -5 & -4 \end{bmatrix} e^{2t} + \begin{bmatrix} -4 & -4 \\ 5 & 5 \end{bmatrix} e^{3t} \\ &= \begin{bmatrix} 5e^{2t} - 4e^{3t} & 4e^{2t} - 4e^{3t} \\ -5e^{2t} + 5e^{3t} & -4e^{2t} + 5te^{2t} \end{bmatrix}. \end{aligned}$$

Then the solution of the initial value problem is given by

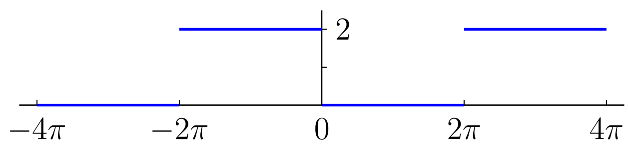
$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = e^{At} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 8e^{2t} - 8e^{3t} \\ -8e^2t + 10e^{3t} \end{bmatrix}.$$

◀

11. [**12 Points**]Let $f(t)$ be the periodic function of period 4π that is defined on the interval $(-2\pi,\, 2\pi]$ by

$$f(t) = \begin{cases} 2 & \text{if } -2\pi < t \le 0, \\ 0 & \text{if } 0 < t \le 2\pi. \end{cases}$$

(a) Sketch the graph of $f(t)$ on the interval $[-4\pi,\, 4\pi]$.



(b) Compute the Fourier series of $f(t)$.

▶ **Solution.** Since the period is 4π, the half-period is 2π so the Fourier series has the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n}{2}t + b_n \sin \frac{n}{2}t)$$

where the coefficients are computed by:

$$a_0 = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(t)\, dt = \frac{1}{2\pi} \int_{-2\pi}^{0} 2\, dt + \frac{1}{2\pi} \int_{0}^{2\pi} 0\, dt = \frac{1}{2\pi} 4\pi = 2.$$

For $n \ge 1$

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_{-2\pi}^{2\pi} f(t) \cos \frac{n}{2}t\, dt \\ &= \frac{1}{2\pi} \int_{-2\pi}^{0} 2 \cos \frac{n}{2}t\, dt \\ &= \frac{2}{n\pi} \sin \frac{n}{2}t \Big|_{-2\pi}^{0} \\ &= 0. \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{2\pi}\int_{-2\pi}^{2\pi} f(t)\sin\frac{n}{2}t\,dt \\ &= \frac{1}{2\pi}\int_{-2\pi}^{0} 2\sin\frac{n}{2}t\,dt \\ &= \frac{1}{2\pi}\cdot\frac{-4}{n}\cos\frac{n}{2}t\Big|_{-2\pi}^{0} \\ &= -\frac{2}{n\pi}(1-\cos n\pi) \\ &= -\frac{2}{n\pi}(1-(-1)^n). \end{aligned}$$

Thus, the Fourier series of $f(t)$ is

$$f(t) \sim 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n}\sin\frac{n}{2}t.$$

◀

(c) Let $g(t)$ denote the sum of the Fourier series found in part (b). Compute $g(2\pi)$ and $g(3\pi)$.

▶ **Solution.** $g(2\pi) = 1$ and $g(3]pi) = 2$. ◀

12. [**12 Points**] A 400-gallon tank initially contains 100 gallons of brine with a concentration of 1.5 ounces of salt per gallon. Starting at time $t = 0$, brine with a salt concentration of 2.0 ounces of salt per gallon runs into the tank at the rate of 6 gallons per minute. The well-mixed solution is drawn off at the same rate of 6 gallons per minute. Let $y(t)$ denote the number of ounces of salt in the tank at time t.

(a) What is $y(0)$?

▶ **Solution.** $y(0) = 100 \times 1.5 = 150$ ounces. ◀

(b) What is the differential equation that $y(t)$ satisfies?

▶ **Solution.**

$$\begin{aligned} y' &= \text{rate in} - \text{rate out} \\ &= 2 \text{ oz/gal}\cdot 6 \text{ gal/min} - \frac{y(t)}{100}\text{ oz/gal}\cdot 6\text{ gal/min}. \end{aligned}$$

Thus, the differential equation is

$$\boxed{y' + \frac{3}{50}y = 12.}$$

◀

(c) Solve this differential equation to determine the amount of salt in the tank at time t.

▶ **Solution.** This is a first order linear differential equation, so use an integrating factor. In this case $\mu(t) = e^{\frac{3}{50}t}$. Multiplying the equation by $\mu(t)$ gives the equation

$$\left(e^{3t/50}y\right)' = 12e^{3t/50}.$$

Integrate to get

$$e^{3t/50}y = 200e^{3t/50} + C$$

and solve for y to get

$$y = 200 - 50e^{-3t/50}.$$

Using the initial condition gives

$$150 = y(0) = 200 + C,$$

which gives $C = -50$ and hence,

$$\boxed{y(t) = 200 - 50e^{-3t/50}.}$$

◀

(d) Find the amount of salt in the tank after 20 minutes.

▶ **Solution.**

$$y(20) = 200 - 50e^{-60/50} = 200 - 503^{-1.2} \approx 184.94 \text{ oz.}$$

◀

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s),$ or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}.$
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s).$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$