Instructions. Answer each of the questions on your own paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* Put your name on each page of your paper.

1. [**8 Points**] Short Answer. For True/False questions you must write 'True' or 'False' and justify your answer.

   (a) True or False: The differential equation $y' = e^{t^2+\sin(y)}$ is separable.

   ▶ **Solution. True**. $y' = e^{t^2+\sin(y)} = e^{t^2}e^{\sin(y)} = h(t)g(y)$ so it is separable. ◀

   (b) True or False: The differential equation $y' + y^2 t = t^2$ is first order linear.

   ▶ **Solution. False**. The y^2 term makes the equation nonlinear. ◀

   (c) What are the possible number of solutions to a system of linear equations?

   ▶ **Solution.** There can be 0 solutions, 1 solution, or infinitely many solutions. ◀

   (d) Let A be a 2015×2015 matrix. If the equation $A\mathbf{x} = \mathbf{0}$ has only the solution $\mathbf{x} = \mathbf{0}$, what do we know about $\det A$?

   ▶ **Solution.** $\det A \neq 0$ ◀

2. [**16 Points**] Find all solutions to the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & x_3 & + & x_4 & = & 12 \\ & & x_2 & - & x_3 & + & 4x_4 & = & 5 \\ 3x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & = & 31 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix}.$$

Row reduce this matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix} \xrightarrow{t_{1\,3}(-3)} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{bmatrix}$$
$$\xrightarrow{t_{2\,3}(1)} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{t_{2\,1}(-1)} \begin{bmatrix} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This last matrix is the augmented matrix of the system

$$\begin{array}{rcrcrcrcr} x_1 & & & + & 2x_3 & - & 3x_4 & = & 7 \\ & & x_2 & - & x_3 & + & 4x_4 & = & 5 \\ & & & & & - & 0 & = & 0 \end{array}$$

This system has leading variables x_1 and x_2 and free variables x_3 and x_4. Solving for x_1 and x_2 in terms of x_3 and x_4, and letting $x_3 = s$, $x_4 = t$, where s and t are arbitrary, gives the solutions

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7-2s+3t \\ 5+s-4t \\ s \\ t \end{bmatrix}, \quad \text{where } s,\ t \text{ are arbitrary.}$$

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3. [**16 Points**] Let $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$.

(a) Compute the inverse of A.

▶ **Solution.**

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -3 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Thus,

$$A^{-1} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$$

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(b) *Using your answer to part* (a), solve the linear system $A\mathbf{x} = \mathbf{b}$ if $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

▶ **Solution.**

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} -3 & 1 & 0 \\ -2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -5 \\ 3 \end{bmatrix}.$$

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4. [**12 Points**]

(a) If A is a 3×3 matrix with $\det A = 3$ and B is obtained from A by first interchanging rows 1 and 2 and then multiplying row 3 by 5, what is $\det(2A^2B)$?

▶ **Solution.** $\det B = (-1)\cdot 5\cdot \det A = -15$. Then

$$\det(2A^2B) = 2^3(\det A)^2 \det B = 8\cdot 9\cdot(-15) = -1080.$$

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(b) Find all values of x for which C^{-1} fails to exist, where $C = \begin{bmatrix} 1 & 2 & 5x \\ 2x-1 & 0 & 0 \\ 3x-5 & 2 & 10 \end{bmatrix}$.

▶ **Solution.** C^{-1} fails to exist when $\det C = 0$. Using cofactor expansion along the second row, $\det C = -(2x-1)\begin{vmatrix} 2 & 5x \\ 2 & 10 \end{vmatrix} = -(2x-10)(20-10x)$. Thus, $\det C = 0$ if and only if $x = 1/2$ or $x = 2$. ◀

5. [**16 Points**] Solve the initial value problem: $y' = e^{2t}y^2, \quad y(0) = -1$.

▶ **Solution.** This equation is separable. There is one equilibrium solution: $y = 0$. This does not satisfy the initial condition so it is not the required solution. If $y \neq 0$, separate variables to get $y^{-2}y' = e^{2t}$, which in differential form is

$$y^{-2}\,dy = e^{2t}, dt$$

and integration then gives

$$-y^{-1} = \int e^{2t}\,dt = \frac{1}{2}e^{2t} + C,$$

so that

$$y = \frac{-1}{\frac{1}{2}e^{2t} + C}.$$

The initial condition $y(0) = -1$ gives $-1 = -1/(\frac{1}{2} + C)$ so $C = \frac{1}{2}$, and

$$\boxed{y = \frac{-1}{\frac{1}{2}e^{2t} + \frac{1}{2}} = \frac{-2}{e^{2t}+1}.}$$

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6. [**16 Points**] Solve the initial value problem: $y' + 4y = 6e^{-4t} + 4e^{-6t}, \quad y(0) = 5$.

▶ **Solution.** This equation is linear with coefficient function $p(t) = 4$ so that an integrating factor is given by $\mu(t) = e^{\int 4\,dt} = e^{4t}$. Multiplication of the differential equation by the integrating factor gives

$$e^{4t}y' + 4e^{4t}y = 6 + 4e^{-2t},$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(e^{4t}y) = 6 + 4e^{-2t}.$$

Integration then gives

$$e^{4t}y = 6t - 2e^{-2t} + C,$$

where C is an integration constant. Multiplying through by e^{-4t} gives

$$y = 6te^{-4t} - 2e^{-6t} + Ce^{-4t}.$$

The initial condition gives $5 = y(0) = -2 + C$ so $C = 7$ and thus

$$\boxed{y(t) = 6te^{-4t} - 2e^{-6t} + 7e^{-4t}.}$$

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7. [**16 Points**] Find the general solution to the differential equation: $y' - 4ty = 2t$.

▶ **Solution.** This equation is linear with coefficient function $p(t) = -4t$ so that an integrating factor is given by $\mu(t) = e^{\int -4t\,dt} = e^{-2t^2}$. Multiplication of the differential equation by the integrating factor gives

$$e^{-2t^2}y' - 4te^{-2t^2}y = 2te^{-2t^2},$$

and the left hand side is recognized (by the choice of $\mu(t)$) as a perfect derivative:

$$\frac{d}{dt}(e^{-2t^2}y) = 2te^{-2t^2}.$$

Integration then gives

$$e^{-2t^2}y = -\frac{1}{2}e^{-2t^2} + C,$$

where C is an integration constant. Multiplying through by e^{2t^2} gives

$$\boxed{y = -\frac{1}{2} + Ce^{2t^2}.}$$

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