Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{array}{rcl}\sin(\theta+\varphi) & = & \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) & = & \cos\theta\cos\varphi - \sin\theta\sin\varphi\end{array}}$$

1. [**18 Points**] Find the the general solution of the following Cauchy-Euler equations:

   (a) $3t^2y'' - 5ty' + 4y = 0$.

   (b) $t^2y'' + 7ty' + 25y = 0$.

2. [**18 Points**] Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

$$t^2y'' - 4ty' + 6y = t^4.$$

   You may assume that the solution of the homogeneous equation $t^2y'' - 4ty' + 6y = 0$ is $y_h = c_1t^2 + c_2t^3$.

3. [**18 Points**] Let f be the function defined by

$$f(t) = \begin{cases} 9t - 2 & \text{if } 0 \le t < 4, \\ 2t^2 & \text{if } t \ge 4. \end{cases}$$

   (a) Express $f(t)$ in terms of translates $h(t-c)$ of the Heaviside unit step function $h(t)$.

   (b) Find the Laplace transform of $f(t)$.

4. [**16 Points**] Find the inverse Laplace transform of the following function:

$$F(s) = e^{-3s}\frac{-5s+2}{s^2+4s+20}$$

5. [**20 Points**]

   (a) Solve the following initial value problem:

$$y'' + 9y = 6h(t-\pi), \qquad y(0) = 0,\ y'(0) = 0.$$

   (Remember that $h(t-c)$ is the Heaviside function (unit step function).)

   (b) Evaluate $y\left(\frac{\pi}{2}\right)$.

   (c) Evaluate $y(2\pi)$.

6. [**10 Points**]

   (a) A vertical spring is stretched 0.1 m downward after a mass of $\frac{1}{2}$ kg is attached to it. The system has a damping constant of $\mu = 10$ and there is no external force acting on the system. Write a differential equation that describes the motion of the spring. Specifically, give a differential equation for $y(t)$, the distance of the mass below the equilibrium point. For simplicity, assume the acceleration due to gravity is $g = 10$ m/s^2. *You are not asked to solve the equation.*

   (b) Is the mass-spring system described in part (a) underdamped, overdamped, or critically damped?

   (c) Determine the value(s) of the damping constant μ for which every solution of the mass-spring system described by the following equation will cross the equilibrium position **no more than once.**

$$5y'' + \mu y' + 20y = 0$$

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$*. Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \textit{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$*. Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \textit{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$