Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms has been appended to the exam. The following trigonometric identities may also be of use:

$$\boxed{\begin{aligned}\sin(\theta+\varphi) &= \sin\theta\cos\varphi + \sin\varphi\cos\theta \\ \cos(\theta+\varphi) &= \cos\theta\cos\varphi - \sin\theta\sin\varphi\end{aligned}}$$

1. **[18 Points]** Find the the general solution of the following Cauchy-Euler equations:

   (a) $3t^2y'' - 5ty' + 4y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = 3s(s-1) - 5s + 4 = 3s^2 - 8s + 4 = (3s-2)(s-2),$$

   which has the two distinct real roots $2/3$ and 2. Hence the general solution is

   $$\boxed{y = c_1t^{2/3} + c_2t^2.}$$

   ◀

   (b) $t^2y'' + 7ty' + 25y = 0$.

   ▶ **Solution.** The indicial polynomial is

   $$Q(s) = s(s-1) + 7s + 25 = s^2 + 6s + 25 = (s+3)^2 + 16,$$

   which has the complex roots $-3 \pm 4i$. Hence the general solution is

   $$\boxed{y = c_1t^{-3}\cos(4\ln|t|) + c_2t^{-3}\sin(4\ln|t|).}$$

   ◀

2. **[18 Points]** Use *variation of parameters* to find a particular solution of the nonhomogeneous differential equation

   $$t^2y'' - 4ty' + 6y = t^4.$$

   You may assume that the solution of the homogeneous equation $t^2y'' - 4ty' + 6y = 0$ is $y_h = c_1t^2 + c_2t^3$.

   ▶ **Solution.** Letting $y_1 = t^2$ and $y_2 = t^3$, a particular solution has the form

   $$y_p = u_1y_1 + u_2y_2 = u_1t^2 + u_2t^3,$$

where u_1 and u_2 are unknown functions whose derivatives satisfy the simultaneous equations

$$u_1' t^2 + u_2' t^3 = 0$$
$$2u_1' t + 3u_2' t^2 = t^2.$$

Applying Cramer's rule gives

$$u_1' = \frac{\begin{vmatrix} 0 & t^3 \\ t^2 & 3t^2 \end{vmatrix}}{\begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}} = \frac{-t^5}{t^4} = -t$$

and

$$u_2' = \frac{\begin{vmatrix} t^2 & 0 \\ 2t & t^2 \end{vmatrix}}{\begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}} = \frac{t^4}{t^4} = 1$$

Integrating gives $u_1 = -t^2/2$ and $u_2 = t$ so that

$$y_p = (-t^2/2)t^2 + tt^3.$$

Hence

$$\boxed{y_p = t^4/2.}$$

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3. **[18 Points]** Let f be the function defined by

$$f(t) = \begin{cases} 9t - 2 & \text{if } 0 \le t < 4, \\ 2t^2 & \text{if } t \ge 4. \end{cases}$$

(a) Express $f(t)$ in terms of translates $h(t-c)$ of the Heaviside unit step function $h(t)$.

▶ **Solution.**

$$\begin{aligned} f(t) &= (9t-2)\chi_{[0,\,4)}(t) + 2t^2\chi_{[4,\,\infty)}(t) \\ &= (9t-2)(h(t) - h(t-4)) + 2t^2 h(t-4) \\ &= (9t-2) + (2t^2 - 9t + 2)h(t-4). \end{aligned}$$

◀

(b) Find the Laplace transform of $f(t)$.

▶ **Solution.** Using the second translation principle,

$$\begin{aligned} F(s) &= \frac{9}{s^2} - \frac{2}{s} + e^{-4s}\mathcal{L}\left\{2(t+4)^2 - 9(t+4) + 2\right\} \\ &= \frac{9}{s^2} - \frac{2}{s} + e^{-4s}\mathcal{L}\left\{2t^2 + 7t - 2\right\} \\ &= \frac{9}{s^2} - \frac{2}{s} + e^{-4s}\Big(\frac{4}{s^3} + \frac{7}{s^2} - \frac{2}{s}\Big). \end{aligned}$$

◀

4. [**16 Points**] Find the inverse Laplace transform of the following function:

$$F(s) = e^{-3s}\frac{-5s+2}{s^2+4s+20}$$

▶ **Solution.** Let $G(s) = \dfrac{-5s+2}{s^2+4s+20}$. Then

$$\begin{aligned} G(s) = \frac{-5s+2}{s^2+4s+20} &= \frac{-5s+2}{(s+2)^2+16} \\ &= \frac{-5(s+2)+12}{(s+2)^2+16} \\ &= \frac{-5(s+2)}{(s+2)^2+16} + \frac{12}{(s+2)^2+20}. \end{aligned}$$

Thus, taking the inverse Laplace transform gives

$$g(t) = \mathcal{L}^{-1}\left\{G(s)\right\} = -5e^{-2t}\cos 4t + 3e^{-2t}\sin 4t,$$

so that the inverse of the second translation theorem gives

$$\begin{aligned} f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} &= g(t-3)h(t-3) \\ &= \Big(-5e^{-2(t-3)}\cos 4(t-3) + 3e^{-2(t-3)}\sin 4(t-3)\Big)h(t-3) \\ &= \begin{cases} 0 & \text{if } 0 \le t < 3 \\ -5e^{-2(t-3)}\cos 4(t-2) + 3e^{-2(t-3)}\sin 4(t-3) & \text{if } t \ge 3. \end{cases} \end{aligned}$$

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5. [**20 Points**]

(a) Solve the following initial value problem:

$$y'' + 9y = 6h(t-\pi), \qquad y(0) = 0,\ y'(0) = 0.$$

(Remember that $h(t-c)$ is the Heaviside function (unit step function).)

▶ **Solution.** Let $Y(s) = \mathcal{L}\{y(t)\}$ be the Laplace transform of the solution function. Apply the Laplace transform to both sides of the equation to get

$$s^2Y(s) + 9Y(s) = 6\frac{e^{-\pi s}}{s}.$$

Thus,

$$Y(s) = e^{-\pi s}\frac{6}{s(s^2+9)}.$$

Use partial fraction decomposition to write

$$\frac{6}{s(s^2+9)} = \frac{2}{3}\Big(\frac{1}{s} - \frac{s}{s^2+9}\Big).$$

Apply the inverse Laplace transform to get

$$\begin{aligned} y(t) &= \frac{2}{3}(1-\cos 3(t-\pi))h(t-\pi) \\ &= \frac{2}{3}(1+\cos 3t)h(t-\pi) \\ &= \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ \frac{2}{3}(1+\cos 3t) & \text{if } t \ge \pi. \end{cases} \end{aligned}$$

◀

(b) Evaluate $y\big(\frac{\pi}{2}\big)$.

▶ **Solution.** $y\big(\frac{\pi}{2}\big) = 0$ ◀

(c) Evaluate $y(2\pi)$.

▶ **Solution.** $y(2\pi) = \frac{2}{3}(1+\cos 6\pi) = \frac{4}{3}$. ◀

6. [**10 Points**]

(a) A vertical spring is stretched 0.1 m downward after a mass of $\frac{1}{2}$ kg is attached to it. The system has a damping constant of $\mu = 10$ and there is no external force acting on the system. Write a differential equation that describes the motion of the spring. Specifically, give a differential equation for $y(t)$, the distance of the mass below the equilibrium point. For simplicity, assume the acceleration due to gravity is $g = 10$ m/s^2. *You are not asked to solve the equation.*

▶ **Solution.** The spring constant is computed from the equation $mg = ku$ where u is the amount the spring is stretched by the mass m. Since $m = \frac{1}{2}$ and $u = 0.1$ this gives $k = mg/u = \frac{\frac{1}{2}\cdot 10}{0.1} = 50$. Thus, the differential equation of motion is

$$\frac{1}{2}y'' + 10y' + 50y = 0.$$

◀

(b) Is the mass-spring system described in part (a) underdamped, overdamped, or critically damped?

▶ **Solution.** The system is critically damped since the discriminant is $D = 10^2 - 4 \cdot \frac{1}{2} \cdot 50 = 0$. ◀

(c) Determine the value(s) of the damping constant μ for which every solution of the mass-spring system described by the following equation will cross the equilibrium position **no more than once.**

$$5y'' + \mu y' + 20y = 0$$

▶ **Solution.** The solutions will pass at most once through the equilibrium if the system is underdamped or critically damped. That is, if the discriminant $D \geq 0$. This will happen when $\mu^2 - 4 \cdot 5 \cdot 20 \geq 0$, that is when $\mu \geq 20$. ◀

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s-\lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$