Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms and some Fourier series and integration formulas have been appended to the exam.

1. Let $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$.

   (a) [**3 Points**] Verify that the characteristic polynomial of A is $c_A(s) = s^2$.

   ▶ **Solution.**

   $$\begin{aligned} c_A(s) = \det(sI - A) &= \det \begin{bmatrix} s-2 & -4 \\ 1 & s+2 \end{bmatrix} = (s-2)(s+2) - (-4)(1) \\ &= s^2 + 2s - 2s - 4 + 4 = s^2. \end{aligned}$$

   ◀

   (b) [**10 Points**] Compute the matrix exponential e^{At}.

   ▶ **Solution.** Use Fulmer's method. The only root of $c_A(s)$ is 0 with multiplicity 2 so $\mathcal{B}_{c_A(s)} = \{1, t\}$. Thus, $e^{At} = M_1 + M_2 t$ for constant matrices M_1 and M_2. Differentiation gives $Ae^{At} = M_2$, and evaluating both equations at $t = 0$ gives

   $$\begin{aligned} I &= M_1 \\ A &= M_2 \end{aligned}$$

   Thus,

   $$\begin{aligned} e^{At} &= I + At \\ &= \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}. \end{aligned}$$

   ◀

   (c) [**7 Points**] Find the general solution of $\mathbf{y}' = A\mathbf{y}$.

   ▶ **Solution.**

   $$\mathbf{y}(t) = e^{At}\mathbf{y}(0) = e^{At}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 + (2c_2 + 4c_2)t \\ c_2 - (c_1 + 2c_2)t \end{bmatrix},$$

   where c_1, c_2 are arbitrary constants. ◀

   (d) [**10 Points**] Solve the initial value problem $\mathbf{y}' = A\mathbf{y} + \mathbf{f}(t)$, $\mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $\mathbf{f}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

▶ **Solution.**

$$\begin{aligned}\mathbf{y}(t) &= e^{At}\begin{bmatrix}0\\0\end{bmatrix} + \int_0^t e^{A(t-u)t}\mathbf{f}(u)\,du\\ &= \begin{bmatrix}0\\0\end{bmatrix} + \int_0^t \begin{bmatrix}1+2(t-u) & 4(t-u)\\ -(t-u) & 1-2(t-u)\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}\,du\\ &= \int_0^t \begin{bmatrix}2+8(t-u)\\ 1-4(t-u)\end{bmatrix}\,du\\ &= \begin{bmatrix}\int_0^t (2+8(t-u))\,du\\ \int_0^t (1-4(t-u))\,du\end{bmatrix}\\ &= \begin{bmatrix}(2u-4(t-u)^2)\big|_0^t\\ (u+2(t-u)^2)\big|_0^t\end{bmatrix} = \begin{bmatrix}2t+4t^2\\ t-2t^2\end{bmatrix}.\end{aligned}$$

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2. Consider the following first order linear system of differential equations

$$\begin{aligned}y_1' &= y_1 - 2y_2\\ y_2' &= 4y_1 - 3y_2\end{aligned}$$

(a) [**3 Points**] Write the system in matrix form $\mathbf{y}' = A\mathbf{y}$.

▶ **Solution.**

$$\mathbf{y}' = \begin{bmatrix}y_1\\y_2\end{bmatrix}' = \begin{bmatrix}1 & -2\\ 4 & -3\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}1 & -2\\ 4 & -3\end{bmatrix}\mathbf{y}.$$

◀

(b) [**3 Points**] Verify that the characteristic polynomial of A is $c_A(s) = (s+1)^2 + 4$.

▶ **Solution.**

$$\begin{aligned}c_A(s) = \det(sI - A) = \det\begin{bmatrix}s-1 & 2\\ -4 & s+3\end{bmatrix} &= (s-1)(s+3) - (-4)(2)\\ &= s^2 + 2s - 3 + 8 = s^2 + 2s + 5 = (s+1)^2 + 4.\end{aligned}$$

◀

(c) [**14 Points**] Solve this system with the initial conditions $y_1(0) = 1$, $y_2(0) = -1$.

▶ **Solution.** First compute e^{At}: $c_A(s) = (s+1)^2 + 4$ so

$$\begin{aligned}(sI - A)^{-1} &= \frac{1}{(s+1)^2+4}\begin{bmatrix}s+3 & -2\\ 4 & s-1\end{bmatrix} = \begin{bmatrix}\frac{s+3}{(s+1)^2+4} & \frac{-2}{(s+1)^2+4}\\ \frac{4}{(s+1)^2+4} & \frac{s-1}{(s+1)^2+4}\end{bmatrix}\\ &= \begin{bmatrix}\frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4} & \frac{-2}{(s+1)^2+4}\\ \frac{4}{(s+1)^2+4} & \frac{s+1}{(s+1)^2+4} + \frac{-2}{(s+1)^2+4}\end{bmatrix}.\end{aligned}$$

Thus,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \begin{bmatrix} e^{-t}\cos 2t + e^{-t}\sin 2t & -e^{-t}\sin 2t \\ 2e^{-t}\sin 2t & e^{-t}\cos 2t - e^{-t}\sin 2t \end{bmatrix},$$

and

$$\begin{aligned} \mathbf{y}(t) &= e^{At}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}\cos 2t + e^{-t}\sin 2t & -e^{-t}\sin 2t \\ 2e^{-t}\sin 2t & e^{-t}\cos 2t - e^{-t}\sin 2t \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} e^{-t}\cos 2t + 2e^{-t}\sin 2t \\ -e^{-t}\cos 2t + 3e^{-t}\sin 2t \end{bmatrix}. \end{aligned}$$

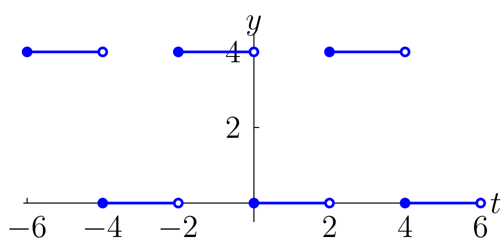
◀

3. Let $f(t)$ be the function of period 4 that is defined on the interval $[-2, 2)$ by

$$f(t) = \begin{cases} 4 & \text{if } -2 \le t < 0, \\ 0 & \text{if } 0 \le t < 2. \end{cases}$$

(a) [**4 Points**] Sketch the graph of $f(t)$ on the interval $[-6, 6]$. Be sure to label the axes of the graph.

▶ **Solution.** ◀



(b) [**3 Points**] Determine all the points $t \in \mathbb{R}$ where $f(t)$ is discontinuous.

▶ **Solution.** $f(t)$ is discontinuous for $0, \pm 2, \pm 4, \pm 6, \cdots$. That is, for all $t = 2n$ for n an integer. ◀

(c) [**3 Points**] Fill in the blanks to complete the convergence theorem for Fourier series:

If $f(t)$ is periodic and piecewise $\boxed{\textbf{smooth}}$*, then its Fourier series converges to*

i. $\boxed{f(t)}$ *at each point t where $f(t)$ is continuous.*

ii. $\boxed{(f(t^+) + f(t^-))/2}$ *at each point t where $f(t)$ is not continuous.*

(d) [**16 Points**] Compute the Fourier series of $f(t)$.

▶ **Solution.** The period is $4 = 2L$ so $L = 2$. Thus,

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\int_{-2}^{0} 4\,dt + \frac{1}{2}\int_{0}^{2} 0\,dt = 4 + 0 = 4.$$

For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\pi}{2}t\,dt \\
&= \frac{1}{2}\int_{-2}^{0} 4\cos\frac{n\pi}{2}t\,dt + \frac{1}{2}\int_{0}^{2} 0\cos\frac{n\pi}{2}t\,dt \\
&= \frac{4}{n\pi}\sin\frac{n\pi}{2}t\Big|_{-2}^{0} \\
&= \frac{4}{n\pi}(\sin 0 - \sin(-n\pi)) \\
&= 0 \quad \text{since } \sin n\pi = 0.
\end{aligned}$$

For $n \geq 1$,

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi}{2}t\,dt \\
&= \frac{1}{2}\int_{-2}^{0} 4\sin\frac{n\pi}{2}t\,dt + \frac{1}{2}\int_{0}^{2} 0\sin\frac{n\pi}{2}t\,dt \\
&= -\frac{4}{n\pi}\cos\frac{n\pi}{2}t\Big|_{-2}^{0} \\
&= -\frac{4}{n\pi}(1 - \cos n\pi) \\
&= \frac{4}{n\pi}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ is even} \\ -\frac{8}{n\pi} & n \text{ is odd.} \end{cases}
\end{aligned}$$

Thus, the Fourier series of $f(t)$ is

$$f(t) \sim 2 - \frac{8}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin\frac{n\pi}{2}t.$$

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(e) [**4 Points**] Let $g(t)$ denote the sum of the Fourier series found in part (d). Compute $g(-4)$ and $g(15.5)$.

▶ **Solution.** $g(-4) = (f(-4^+) + f(-4^-))/2 = (0+4)/2 = 2$ since f has a jump discontinuity at -4. $g(t)$ is periodic of period 4 so $g(15.5) = g(15.5 - 3\cdot 4) = g(3.5) = f(3.5) = 4$, since f is continuous at 3.5. ◀

4. Consider the function

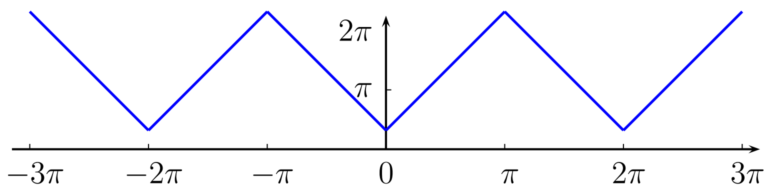
$$f(t) = 2t + 1, \qquad 0 < t < \pi.$$

(a) **[4 Points]** Let $f_e(t)$ be the even periodic extension of $f(t)$. Give an explicit formula for $f_e(t)$ and draw the graph of $f_e(t)$ on the interval $[-3\pi, 3\pi]$. Be sure to label the graph.

▶ **Solution.** The even periodic extension is

$$f_e(t) = \begin{cases} f(t) & \text{if } 0 < t < \pi \\ f(-t) & \text{if } -\pi < t < 0 \end{cases} = \begin{cases} 2t + 1 & \text{if } 0 < t < \pi \\ -2t + 1 & \text{if } -pi < t < 0 \end{cases}, \quad f(t+2\pi) = f(t).$$

The graph is:



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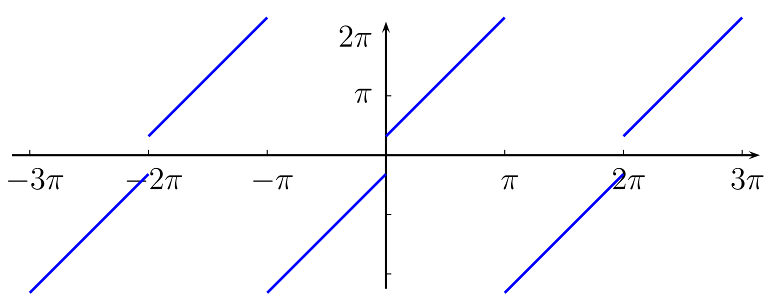
(b) **[4 Points]** Let $f_o(t)$ be the odd periodic extension of $f(t)$. Give an explicit formula for $f_o(t)$ and draw the graph of $f_o(t)$ on the interval $[-3\pi, 3\pi]$.

▶ **Solution.** The odd periodic extension is

$$f_0(t) = \begin{cases} f(t) & \text{if } 0 < t < \pi \\ -f(-t) & \text{if } -\pi < t < 0 \end{cases} = \begin{cases} 2t + 1 & \text{if } 0 < t < \pi \\ 2t - 1 & \text{if } -pi < t < 0 \end{cases}, \quad f(t+2\pi) = f(t).$$

The graph is:



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(c) **[12 Points]** Compute the Fourier cosine series of $f(t)$.

▶ **Solution.** The cosine series has only cosine terms. For $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi}{L}t\,dt \\
&= \frac{2}{\pi}\int_0^2 (2t+1)\cos nt\,dt \\
&= \frac{2}{\pi}\left[\frac{2}{n^2}\cos nt - \frac{2t+1}{n}\sin nt\right]_0^\pi \\
&= \frac{4}{n^2\pi}(\cos n\pi - 1) \\
&= \frac{4}{n\pi}((-1)^n - 1) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{8}{n^2\pi} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

For $n = 0$,

$$a_0 = \frac{2}{\pi}\int_0^\pi (2t+1)\,dt = \frac{2}{\pi}\left.(t^2+t)\right|_0^\pi = 2(\pi+1).$$

Thus, the Fourier cosine series is

$$f(t) \sim (\pi+1) - \sum_{n=\text{odd}} \frac{8}{n^2\pi}\cos nt.$$

◀

The following Fourier series and integration formulas may be of use:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^\infty \left(a_n\cos\frac{n\pi}{L}t + b_n\sin\frac{n\pi}{L}t\right) \quad \text{where}$$

$$a_n = \frac{1}{L}\int_{-L}^L f(t)\cos\frac{n\pi}{L}t\,dt, \qquad b_n = \frac{1}{L}\int_{-L}^L f(t)\sin\frac{n\pi}{L}t\,dt.$$

$$\begin{aligned}
\int (bt+c)\sin at\,dt &= \frac{b}{a^2}\sin at - \frac{bt+c}{a}\cos at \\
\int (bt+c)\cos at\,dt &= \frac{b}{a^2}\cos at + \frac{bt+c}{a}\sin at.
\end{aligned}$$

Laplace Transform Table

	$f(t)$	$\longleftrightarrow$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\longleftrightarrow$	$\dfrac{1}{s}$
2.	t^n	$\longleftrightarrow$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\longleftrightarrow$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\longleftrightarrow$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\longleftrightarrow$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\longleftrightarrow$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\longleftrightarrow$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\longleftrightarrow$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\longleftrightarrow$	$\dfrac{e^{-sc}}{s}$
10.	$\delta_c(t) = \delta(t-c)$	$\longleftrightarrow$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.