Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, the definition of Fourier series, and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**10 Points**] $y' = e^{2t}/y, \quad y(0) = -3.$

2. [**10 Points**] $ty' - 3y = t, \quad y(4) = -1.$

3. [**10 Points**] $2y'' + 2y' + 5y = 0.$

4. [**10 Points**] $4y'' + 5y = 0, \quad y(0) = 2,\ y'(0) = 5.$

5. [**10 Points**] $y'' - 6y' + 9y = \delta(t-1) - 2\delta(t-3), \quad y(0) = 0,\ y'(0) = 0.$ Recall that $\delta(t-c))$ refers to the Dirac delta function providing a unit impulse at time c.

6. [**10 Points**] $y'' - 6y' - 7y = 8e^{-t} - 7t - 6.$

7. [**10 Points**] Find a particular solution for $t > 0$ of the differential equation
$$y'' - 2y' + y = t^{-3}e^t,$$
given that the general solution of the associated homogeneous equation is
$$y_h(t) = c_1e^t + c_2te^t.$$

8. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:
$$f(t) = \begin{cases} t-3 & \text{if } 0 \le t < 3 \\ t^2+1 & \text{if } 3 \le t < 5 \\ 1 & \text{if } t \ge 5. \end{cases}$$

9. [**10 Points**] Compute the inverse Laplace transform of the following function:
$$F(s) = \frac{2s^2 - 10s - 18}{s^3 - s^2 - 6s}$$

10. [**10 Points**] Solve the initial value problem:
$$\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

11. [**10 Points**] The periodic function $f(t)$ is defined in one full period by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ -1 & \text{if } -2 \le t < 0. \end{cases}$$

   (a) Find the Fourier series of $f(t)$.

   (b) What is the sum of the Fourier series at $t = 1$?

   (c) Using your answer to part (b), find a formula expressing π as a sum of an infinite series.

12. [**10 Points**] A mass spring system has mass $m = 2$, spring constant $k = 1$, damping constant $\mu = 3$, and no externally applied force.

   (a) If $y(t)$ denotes the displacement from the equilibrium position, what is the differential equation satisfied by $y(t)$?

   (b) If $y(0) = 2$ and $y'(0) = 0$, find $y(t)$.

   (c) Is the system over damped, under damped, or critically damped?

Bonus. [**10 Points**] Find the solution of the heat conduction problem

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0$$
$$u(0,\, t) = 0, \quad u(4,\, t) = 0 \quad t > 0$$
$$u(x,\, 0) = f(x) = 2\sin\tfrac{\pi}{4}x - \sin \pi x, \quad 0 < x < 4.$$

Recall that the solution of the general heat conduction problem is given by

$$u(x,\, t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2}{L^2}a^2 t} \sin\frac{n\pi}{L}x.$$

Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\dfrac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series and is:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t\right) \quad \text{where}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi}{L} t \, dt, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L} t \, dt.$$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. Then there is a unique number A_1 and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 and the polynomial $p_1(s)$ are given by

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s - \lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d is an irreducible quadratic that is factored completely out of $q(s)$. Then there is a unique linear term $B_1 s + C_1$ and a unique polynomial $p_1(s)$ such that

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ is a complex root of s^2+cs+d then $B_1 s + C_1$ and the polynomial $p_1(s)$ are given by

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$