Instructions. Answer each of the questions on your own paper. Put your name on each page of your paper. Be sure to show your work so that partial credit can be adequately assessed. *Credit will not be given for answers (even correct ones) without supporting work.* A table of Laplace transforms, the definition of Fourier series, and the statement of the partial fraction decomposition theorems are attached to the exam.

In Exercises 1 – 6, solve the given differential equation. If initial values are given, solve the initial value problem. Otherwise, give the general solution. Some problems may be solvable by more than one technique. You are free to choose whatever technique that you deem to be most appropriate.

1. [**10 Points**] $y' = e^{2t}/y, \quad y(0) = -3$.

   ▶ **Solution.** This equation is separable. Separate the variables to get $yy' = e^{2t}$. Write in differential form and integrate to get:

$$\int y\,dy = \int e^{2t}\,dt.$$

   Integrating gives $\frac{1}{2}y^2 = \frac{1}{2}e^{2t} + C$, and multiplying by 2 gives $y^2 = e^{2t} + B$ where $B = 2C$ is a constant. The initial condition $y = -3$ when $t = 0$ gives $9 = 1 + B$ so that $B = 8$. Hence, $y(t) = \pm\sqrt{e^{2t} + 8}$. Since $y(0) < 0$, the negative sign is needed. Thus

$$\boxed{y(t) = -\sqrt{e^{2t} + 8}.}$$

◀

2. [**10 Points**] $ty' - 3y = t, \quad y(4) = -1$.

   ▶ **Solution.** This is a first order linear equation, that must first be put in standard form by dividing by t:

$$y' - \frac{3}{t}y = 1.$$

   Now compute an integrating factor

$$\mu(t) = e^{-\int \frac{3}{t}\,dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3},$$

   and multiply the equation by $\mu(t) = t^{-3}$ to get $t^{-3}y' - 3t^{-4}y = t^{-3}$. The left hand side is $(t^{-3}y)'$ so we get the equation $(t^{-3}y)' = t^{-3}$ and integration then gives $t^{-3}y = -\frac{1}{2}t^{-2} + C$ so that $y(t) = -\frac{1}{2}t + ct^3$. Using the initial condition $y(4) = -1$ gives $-1 = y(4) = -\frac{1}{2}\cdot 4 + C4^3$, which implies $1 = C4^3$ so that $C = \frac{1}{64}$. Hence,

$$\boxed{y(t) = -\frac{1}{2}t + \frac{1}{64}t^3.}$$

◀

3. [**10 Points**] $2y'' + 2y' + 5y = 0$.

▶ **Solution.** The characteristic polynomial is $q(s) = 2s^2 + 2s + 5$. The roots can be calculated from the quadratic formula as

$$\frac{-2 \pm \sqrt{2^2 - 4 \cdot 2 \cdot 5}}{4} = -\frac{1}{2} \pm \frac{3}{2}i.$$

Thus, the general solution is given by

$$\boxed{y(t) = c_1 e^{-t/2} \cos 3t/2 + c_2 e^{-t/2} \sin 3t/2.}$$

◀

4. [**10 Points**] $4y'' + 5y = 0, \quad y(0) = 2,\ y'(0) = 5.$

▶ **Solution.** This is a constant coefficient homogeneous linear equation with characteristic polynomial $q(s) = 4s^2 + 5$ which has roots $\pm\frac{\sqrt{5}}{2}i$. Hence, the general solution is $y = c_1 \cos(\sqrt{5}/2)t + c_2 \sin(\sqrt{5}/2)t$. The initial conditions imply that $2 = y(0) = c_1$ and $5 = y'(0) = (\sqrt{5}/2)c_2$ so that $c_2 = 2\sqrt{5}$. Thus

$$\boxed{y = 2\cos\frac{\sqrt{5}t}{2} + 2\sqrt{5}\sin\frac{\sqrt{5}t}{2}.}$$

◀

5. [**10 Points**] $y'' - 6y' + 9y = \delta(t-1) - 2\delta(t-3), \quad y(0) = 0,\ y'(0) = 0.$ Recall that $\delta(t-c))$ refers to the Dirac delta function providing a unit impulse at time c.

▶ **Solution.** Use the Laplace transform method. Let $Y(s) = \mathcal{L}\{y(t)\}$ where $y(t)$ is the unknown solution of the initial value problem. Applying the Laplace transform to the differential equation gives:

$$s^2Y(s) - 6sY(s) + 9Y(s) = e^{-s} - 2e^{-3s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{e^{-s}}{(s-3)^2} - 2\frac{e^{-3s}}{(s-3)^2}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = te^{3t} = f(t),$$

taking the inverse Laplace transform of $Y(s)$ using the inverse of the second translation theorem gives:

$$\begin{aligned} y(t) = \mathcal{L}^{-1}\{Y(s)\} &= h(t-1)f(t-1) - 2h(t-3)f(t-3) \\ &= h(t-1)((t-1)e^{3(t-1)}) - 2h(t-3)((t-3)e^{3(t-3)}). \end{aligned}$$

◀

6. [**10 Points**] $y'' - 6y' - 7y = 8e^{-t} - 7t - 6$.

   ▶ **Solution.** Use the method of undetermined coefficients. The characteristic polynomial is $q(s) = s^2 - 6s - 7 = (s-7)(s+1)$ which has roots 7 and -1. Thus $\mathcal{B}_q = \{e^{7t}, e^{-t}\}$ and $y_h = c_1 e^{7t} + c_2 e^{-t}$. Since

$$\mathcal{L}\left\{8e^{-t} - 7t - 6\right\} = \frac{8}{s+1} - \frac{7}{s^2} - \frac{6}{s} = \frac{8s^2 - 7(s+1) - 6s(s+1)}{s^2(s+1)}$$

   the denominator is $v = s^2(s+1)$ and $qv = (s-7)(s+1)^2 s^2$. Hence,

$$\mathcal{B}_{qv} \setminus \mathcal{B}_q = \left\{e^{7t}, e^{-t}, te^{-t}, 1, t\right\} \setminus \left\{e^{7t}, e^{-t}\right\} = \left\{te^{-t}, 1, t\right\}.$$

   Therefore, the test function for y_p is $y_p = Ate^{-t} + B + Ct$. Compute the derivatives:

$$\begin{aligned} y_p' &= A(e^{-t} - te^{-t}) + C \\ y_p'' &= A(-2e^{-t} + te^{-t}). \end{aligned}$$

   Substituting into the differential equation gives

$$\begin{aligned} 8e^{-t} - 7t - 6 &= y_p'' - 6y_p' - 7y_p \\ &= A(-2e^{-t} + te^{-t}) - 6(A(e^{-t} - te^{-t}) + C) - 7(Ate^{-t} + B + Ct) \\ &= (-8A)e^{-t} - 7Ct - 7B - 6C. \end{aligned}$$

   Comparing the coefficients of e^{-t}, 1, and t on both sides of this equation shows that A, B, and C satisfy the system of linear equations

$$\begin{aligned} -8A &= 8 \\ -7C &= -7 \\ -7B - 6C &= -6. \end{aligned}$$

   Thus $A = -1$, $C = 1$, and $B = 0$. Thus,

$$y_p = -te^{-t} + t,$$

   and

$$y_g = y_h + y_p = c_1 e^{7t} + c_2 e^{-t} - te^{-t} + t.$$

   ◀

7. [**10 Points**] Find a particular solution for $t > 0$ of the differential equation

$$y'' - 2y' + y = t^{-3} e^t,$$

   given that the general solution of the associated homogeneous equation is

$$y_h(t) = c_1 e^t + c_2 te^t.$$

▶ **Solution.** Use variation of parameters. A particular solution is given by

$$y_p = u_1 e^t + u_2 t e^t$$

where u_1' and u_2' satisfy the equations:

$$\begin{aligned} u_1' e^t + u_2' t e^t &= 0 \\ u_1' e^t + u_2'(t+1)e^t &= t^{-3}e^t. \end{aligned}$$

Dividing both equations by e^t gives the equivalent set of equations:

$$\begin{aligned} u_1' + u_2' t &= 0 \\ u_1' + u_2'(t+1) &= t^{-3} \end{aligned}$$

Solving for u_1' and u_2' via Cramer's rule gives

$$u_1' = \frac{\begin{vmatrix} 0 & t \\ t^{-3} & t+1 \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix}} = -t^{-2},$$

and

$$u_2' = \frac{\begin{vmatrix} 1 & 0 \\ 1 & t^{-3} \end{vmatrix}}{\begin{vmatrix} 1 & t \\ 1 & t+1 \end{vmatrix}} = t^{-3},$$

Integrating then gives

$$u_1 = \frac{1}{t} \quad \text{and} \quad u_2 = -\frac{1}{2}t^{-2},$$

which gives

$$y_p = \frac{1}{t}e^t - \frac{1}{2}t^{-2}te^t = \frac{1}{2t}e^t.$$

◀

8. [**10 Points**] Compute the Laplace transform $F(s)$ of the function $f(t)$ defined as follows:

$$f(t) = \begin{cases} t-3 & \text{if } 0 \le t < 3 \\ t^2+1 & \text{if } 3 \le t < 5 \\ 1 & \text{if } t \ge 5. \end{cases}$$

▶ **Solution.** Using characteristic functions, write $f(t)$ in terms of unit step function:

$$\begin{aligned} f(t) &= (t-3)\chi_{[0,\,3)}(t) + (t^2+1)\chi_{[3,\,5)}(t) + \chi_{[5,\,\infty)}(t) \\ &= (t-3)(1-h(t-3)) + (t^2+1)(h(t-3)-h(t-5)) + h(t-5) \\ &= (t-3) + (t^2-t+4)h(t-3) - t^2h(t-5). \end{aligned}$$

By linearity and the second translation theorem

$$\begin{aligned} F(s) = \mathcal{L}\left\{f(t)\right\} &= \mathcal{L}\left\{(t-3) + (t^2 - t + 4)h(t-3) - t^2 h(t-5)\right\} \\ &= \left(\frac{1}{s^2} - \frac{3}{s}\right) + e^{-3s}\mathcal{L}\left\{(t+3)^2 - (t+3) + 4\right\} - e^{-5s}\mathcal{L}\left\{(t+5)^2\right\} \\ &= \left(\frac{1}{s^2} - \frac{3}{s}\right) + e^{-3s}\mathcal{L}\left\{t^2 + 5t + 10\right\} - e^{-5s}\mathcal{L}\left\{t^2 + 10t + 25\right\} \\ &= \left(\frac{1}{s^2} - \frac{3}{s}\right) + e^{-3s}\left(\frac{2}{s^3} + \frac{5}{s^2} + \frac{10}{s}\right) - e^{-5s}\left(\frac{2}{s^3} + \frac{10}{s^2} + \frac{25}{s}\right). \end{aligned}$$

◀

9. [**10 Points**] Compute the inverse Laplace transform of the following function:

$$F(s) = \frac{2s^2 - 10s - 18}{s^3 - s^2 - 6s}$$

▶ **Solution.** Expand $F(s)$ in partial fractions. The denominator factors as

$$s^3 - s^2 - 6s = s(s-3)(s+2),$$

which is a product of distinct linear factors. Thus we can write

$$F(s) = \frac{2s^2 - 10s - 18}{s^3 - s^2 - 6s} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+2},$$

where

$$\begin{aligned} A &= \left.\frac{2s^2 - 10s - 18}{(s-3)(s+2)}\right|_{s=0} = \frac{-18}{(-3)(2)} = 3, \\ B &= \left.\frac{2s^2 - 10s - 18}{s(s+2)}\right|_{s=3} = \frac{18 - 30 - 18}{3(3+2)} = -2, \\ C &= \left.\frac{2s^2 - 10s - 18}{s(s-3)}\right|_{s=-2} = \frac{8 + 20 - 18}{(-2)(-5)} = 1. \end{aligned}$$

Thus,

$$F(s) = \frac{3}{s} - \frac{2}{s-3} + \frac{1}{s+2},$$

so taking the inverse Laplace transform gives

$$f(t) = 3 - 2e^{3t} + e^{-2t}.$$

◀

10. [**10 Points**] Solve the initial value problem:

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

▶ **Solution.** The system is given in matrix form $\mathbf{y}' = A\mathbf{y}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix}.$$

Start by computing e^{At}: Use Fulmer's method. First compute the characteristic polynomial $c_A(s)$:

$$\begin{aligned} c_A(s) = \det(sI - A) &= \det \begin{bmatrix} s-1 & 1 \\ -5 & s+3 \end{bmatrix} = (s-1)(s+3) - (-5)\cdot 1 \\ &= s^2 + 2s - 3 + 5 = (s+1)^2 + 1. \end{aligned}$$

The roots of $c_A(s)$ are $-1 \pm i$ so $\mathcal{B}_{c_A(s)} = \{e^{-t}\cos t,\, e^{-t}\sin t\}$. Thus,

$$e^{At} = M_1 e^{-t}\cos t + M_2 e^{-t}\sin t$$

for constant matrices M_1 and M_2. Differentiation gives

$$Ae^{At} = M_1(-e^{-t}\cos t - e^{-t}\sin t) + M_2(-e^{-t}\sin t + e^{-t}\cos t),$$

and evaluating both equations at $t = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= -M_1 + M_2 \end{aligned}$$

Solving for M_1 and M_2 gives

$$\begin{aligned} M_1 &= I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ M_2 &= (A + I) = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} e^{-t}\cos t + \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} e^{-t}\sin t \\ &= \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t & -e^{-t}\sin t \\ 5e^{-t}\sin t & e^{-t}\cos t - 2e^{-t}\sin t \end{bmatrix}. \end{aligned}$$

Then the solution of the initial value problem is $\mathbf{y}(t) = e^{At}\mathbf{y}(0)$, that is

$$\begin{aligned} \mathbf{y}(t) &= \begin{bmatrix} e^{-t}\cos t + 2e^{-t}\sin t & -e^{-t}\sin t \\ 5e^{-t}\sin t & e^{-t}\cos t - 2e^{-t}\sin t \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t}\cos t + e^{-t}\sin t \\ 3e^{-t}\cos t + 4e^{-t}\sin t \end{bmatrix}. \end{aligned}$$

◀

11. [**10 Points**] The periodic function $f(t)$ is defined in one full period by

$$f(t) = \begin{cases} 1 & \text{if } 0 \le t < 2 \\ -1 & \text{if } -2 \le t < 0. \end{cases}$$

(a) Find the Fourier series of $f(t)$.

▶ **Solution.** One full period is the interval $[-2, 2)$, so the period is 4 and $4 = 2L$ so $L = 2$. The function $f(t)$ is odd, so all the cosine terms $a_n = 0$, and

$$\begin{aligned} b_n &= \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi}{2}t\,dt = \frac{2}{2}\int_0^2 \sin\frac{n\pi}{2}t\,dt \\ &= -\frac{2}{n\pi}\cos\frac{n\pi}{2}t\Big|_0^2 = -\frac{2}{n\pi}(\cos n\pi - 1) \\ &= \frac{2}{n\pi}(1-\cos n\pi) = \frac{2}{n\pi}(1-(-1)^n) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Thus, the Fourier series is

$$f(t) \sim \frac{4}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin\frac{n\pi}{2}t.$$

◀

(b) What is the sum of the Fourier series at $t = 1$?

▶ **Solution.** Since $f(t)$ is continuous at $t = 1$, by the convergence theorem, the Fourier series of $f(t)$ at $t = 1$ converges to $f(1) = 1$. ◀

(c) Using your answer to part (b), find a formula expressing π as a sum of an infinite series.

▶ **Solution.** By part (b), the sum of the Fourier series at $t = 1$ is 1, so

$$\begin{aligned} 1 &= \frac{4}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin\frac{n\pi}{2} \\ &= \frac{4}{\pi}\Big(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \cdots\Big) \\ &= \frac{4}{\pi}\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big). \end{aligned}$$

Solving for π gives

$$\pi = 4\Big(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\Big).$$

◀

12. [**10 Points**] A mass spring system has mass $m = 2$, spring constant $k = 1$, damping constant $\mu = 3$, and no externally applied force.

(a) If $y(t)$ denotes the displacement from the equilibrium position, what is the differential equation satisfied by $y(t)$?

▶ **Solution.** $2y'' + 3y' + y = 0$. ◀

(b) If $y(0) = 2$ and $y'(0) = 0$, find $y(t)$.

▶ **Solution.** The characteristic polynomial is $q(s) = 2s^2+3s+1 = (2s+1)(s+1)$ which has roots $-1/2$ and -1. Thus, $y(t) = c_1e^{-t/2}+C_2e^{-t}$. The initial conditions give

$$\begin{aligned} 2 &= y(0) = c_1 + c_2 \\ 0 &= y'(0) = -\frac{1}{2}c_1 - c_2. \end{aligned}$$

Solving these equations gives $c_1 = 4$, $c_2 = -2$ so

$$\boxed{y(t) = 4e^{-t/2} - 2e^{-t}.}$$

◀

(c) Is the system over damped, under damped, or critically damped?

▶ **Solution.** The discriminant of the characteristic polynomial is $3^2 - 4 \cdot 2 \cdot 1 = 1 > 0$ so the system is over damped. ◀

Bonus. [**10 Points**] Find the solution of the heat conduction problem

$$\begin{gathered} \frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < 4, \quad t > 0 \\ u(0,\,t) = 0, \quad u(4,\,t) = 0 \quad t > 0 \\ u(x,\,0) = f(x) = 2\sin\tfrac{\pi}{4}x - \sin\pi x, \quad 0 < x < 4. \end{gathered}$$

Recall that the solution of the general heat conduction problem is given by

$$u(x,\,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2}{L^2}a^2t}\sin\frac{n\pi}{L}x.$$

▶ **Solution.** The length of the rod is $L = 4$, the heat conduction coefficient is $a^2 = 5$, and the coefficients c_n are the coefficients of the Fourier sine series of $f(x)$. That is, write

$$f(x) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi}{4}x.$$

But $f(x)$ is already given as a (finite) Fourier sine series with coefficients $c_1 = 2$, $c_4 = -1$ and $c_n = 0$ if $n \neq 1, 4$. Thus,

$$u(x, t) = 2e^{-\frac{5\pi^2}{4^2}t} \sin \frac{\pi}{4}x - e^{-5\pi^2 t} \sin \pi x.$$

◀

Laplace Transform Table

	$f(t)$	$\to$	$F(s) = \mathcal{L}\{f(t)\}(s)$
1.	1	$\to$	$\dfrac{1}{s}$
2.	t^n	$\to$	$\dfrac{n!}{s^{n+1}}$
3.	e^{at}	$\to$	$\dfrac{1}{s-a}$
4.	$t^n e^{at}$	$\to$	$\dfrac{n!}{(s-a)^{n+1}}$
5.	$\cos bt$	$\to$	$\dfrac{s}{s^2+b^2}$
6.	$\sin bt$	$\to$	$\dfrac{b}{s^2+b^2}$
7.	$e^{at}\cos bt$	$\to$	$\dfrac{s-a}{(s-a)^2+b^2}$
8.	$e^{at}\sin bt$	$\to$	$\dfrac{b}{(s-a)^2+b^2}$
9.	$h(t-c)$	$\to$	$\dfrac{e^{-sc}}{s}$
10.	$\delta(t-c)$	$\to$	e^{-sc}

Laplace Transform Principles

Linearity	$\mathcal{L}\{af(t)+bg(t)\}$	$=$	$a\mathcal{L}\{f\}+b\mathcal{L}\{g\}$
Input Derivative Principles	$\mathcal{L}\{f'(t)\}(s)$	$=$	$s\mathcal{L}\{f(t)\}-f(0)$
	$\mathcal{L}\{f''(t)\}(s)$	$=$	$s^2\mathcal{L}\{f(t)\}-sf(0)-f'(0)$
First Translation Principle	$\mathcal{L}\{e^{at}f(t)\}$	$=$	$F(s-a)$
Transform Derivative Principle	$\mathcal{L}\{-tf(t)\}(s)$	$=$	$\dfrac{d}{ds}F(s)$
Second Translation Principle	$\mathcal{L}\{h(t-c)f(t-c)\}$	$=$	$e^{-sc}F(s)$, or
	$\mathcal{L}\{g(t)h(t-c)\}$	$=$	$e^{-sc}\mathcal{L}\{g(t+c)\}$.
The Convolution Principle	$\mathcal{L}\{(f*g)(t)\}(s)$	$=$	$F(s)G(s)$.

Fourier Series Definition

For a function $f(t)$ of period $2L$ the Fourier series and is:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t\right) \quad \text{where}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos \frac{n\pi}{L} t \, dt, \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin \frac{n\pi}{L} t \, dt.$$

Partial Fraction Expansion Theorems

The following two theorems are the main partial fractions expansion theorems, as presented in the text.

Theorem 1 (Linear Case). *Suppose a proper rational function can be written in the form*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)}$$

and $q(\lambda) \neq 0$. *Then there is a unique number* A_1 *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s-\lambda)^n q(s)} = \frac{A_1}{(s-\lambda)^n} + \frac{p_1(s)}{(s-\lambda)^{n-1} q(s)}. \tag{1}$$

The number A_1 *and the polynomial* $p_1(s)$ *are given by*

$$A_1 = \frac{p_0(\lambda)}{q(\lambda)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - A_1 q(s)}{s - \lambda}. \tag{2}$$

Theorem 2 (Irreducible Quadratic Case). *Suppose a real proper rational function can be written in the form*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)},$$

where s^2+cs+d *is an irreducible quadratic that is factored completely out of* $q(s)$. *Then there is a unique linear term* $B_1 s + C_1$ *and a unique polynomial* $p_1(s)$ *such that*

$$\frac{p_0(s)}{(s^2+cs+d)^n q(s)} = \frac{B_1 s + C_1}{(s^2+cs+d)^n} + \frac{p_1(s)}{(s^s+cs+d)^{n-1} q(s)}. \tag{3}$$

If $a+ib$ *is a complex root of* s^2+cs+d *then* $B_1 s + C_1$ *and the polynomial* $p_1(s)$ *are given by*

$$B_1(a+ib) + C_1 = \frac{p_0(a+ib)}{q(a+ib)} \qquad \text{and} \qquad p_1(s) = \frac{p_0(s) - (B_1 s + C_1) q(s)}{s^2+cs+d}. \tag{4}$$