Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the Fourier series supplement:

Section 10.2: 2, 10, 12

2. The period is 2π so $L = \pi$. Use the Euler formulas for a_n to conclude

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\, dt = 0$$

for all $n \geq 0$. This is because the function $f(t)\cos\frac{n\pi}{L}t$ is the product of an odd and even function, and hence is odd, which implies that the integral is 0. Now compute the coefficients b_n:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\, dt = \frac{1}{\pi}\int_{-\pi}^{0} f(t)\sin nt\, dt + \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt\, dt \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (-2)\sin nt\, dt + \frac{1}{\pi}\int_{0}^{\pi} (+2)\sin nt\, dt \\
&= \frac{1}{\pi}\left\{\left[\frac{1}{n}\cos nt\right]_{-\pi}^{0} - \left[\frac{1}{n}\cos nt\right]_{0}^{\pi}\right\} \\
&= \frac{2}{n\pi}[(1-\cos(-n\pi)) + (1-\cos(n\pi))] \\
&= \frac{4}{n\pi}(1-\cos n\pi) = \frac{4}{n\pi}(1-(-1)^n).
\end{aligned}$$

Therefore,

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{8}{n\pi} & \text{if } n \text{ is odd,} \end{cases}$$

and the Fourier series is

$$\begin{aligned}
f(t) &\sim \frac{8}{\pi}\left(\sin nt + \frac{1}{3}\sin nt + \frac{1}{5}\sin nt + \frac{1}{7}\sin nt + \cdots\right) \\
&= \frac{8}{\pi}\sum_{n=\text{odd}} \frac{1}{n}\sin nt.
\end{aligned}$$

10. The period is 2π so $L = \pi$ and $n\pi/L = n$. For the cosine terms a_n:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin t\, dt = -\frac{1}{\pi}\cos t\Big|_0^{\pi} = \frac{2}{\pi},$$

and for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{0}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_{0}^{\pi} \sin t\cos nt\,dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}(\sin(n+1)t - \sin(n-1)t)\,dt \\
&= \frac{1}{2\pi}\left[\frac{-1}{n+1}\cos(n+1)t + \frac{1}{n-1}\cos(n-1)t\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\frac{-1}{n+1}(\cos(n+1)\pi - 1) + \frac{1}{n-1}(\cos(n-1)\pi - 1)\right] \\
&= \begin{cases} \frac{-1}{\pi}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] = \frac{-2}{(n^2-1)\pi} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

For the sine terms b_n, first do $n = 1$.

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin t\,dt = \frac{1}{\pi}\int_{0}^{\pi} \sin^2 t\,dt = \frac{1}{2\pi}\int_{0}^{\pi}(1 - \cos 2t)\,dt = \frac{1}{2}.$$

For $n > 1$,

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{0}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_{0}^{\pi} \sin t\sin nt\,dt \\
&= \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}(\cos(n-1)t - \cos(n+1)t)\,dt \\
&= \frac{1}{2\pi}\left[\frac{1}{n-1}\sin(n-1)t - \frac{1}{n+1}\sin(n+1)t\right]_0^{\pi} \\
&= 0
\end{aligned}$$

Therefore, the Fourier series is

$$f(t) \sim \frac{1}{\pi} + \frac{1}{2}\sin t - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nt}{4n^2 - 1}.$$

12. The function $f(t)$ is periodic of period $2 = 2L$ so $L = 1$ and it is odd so the cosine terms $a_n = 0$ for all n. Compute the b_n terms from Euler's formula:

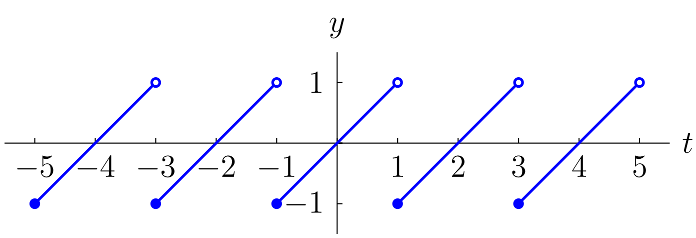
$$\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi}{L}t\,dt = \int_{-1}^{1} f(t)\sin n\pi t\,dt \\
&= 2\int_0^1 (-1+t)\sin n\pi t\,dt = -2\int_0^1 \sin n\pi t\,dt + 2\int_0^1 t\sin n\pi t\,dt \\
&= \frac{2}{n\pi}\cos n\pi t\Big|_0^1 + 2\int_0^1 t\sin n\pi t\,dt \\
&= \frac{2}{n\pi}((-1)^n - 1) + 2\int_0^1 t\sin n\pi t\,dt \qquad \left(\text{let } x = n\pi t \text{ so } t = \frac{1}{n\pi}x \text{ and } dt = \frac{1}{n\pi}dx\right) \\
&= \frac{2}{n\pi}((-1)^n - 1) + 2\int_0^{n\pi} \frac{1}{n\pi}x\sin x\,\frac{1\,dx}{n\pi} = \frac{2}{n^2\pi^2}\int_0^{n\pi} x\sin x\,dx \\
&= \frac{2}{n\pi}((-1)^n - 1) + \frac{2}{n^2\pi^2}\left[\sin x - x\cos x\right]_{x=0}^{x=n\pi} \\
&= \frac{2}{n\pi}((-1)^n - 1) - \frac{2}{n^2\pi^2}(n\pi\cos n\pi) = \frac{2}{n\pi}((-1)^n - 1) - \frac{2}{n\pi}(-1)^n = -\frac{2}{n\pi}.
\end{aligned}$$

Thus, the Fourier series is

$$f(t) \sim -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi t}{n}.$$

Section 10.3: 2, 6, 12

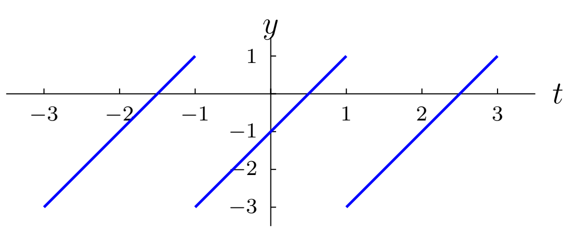
2. (a)



(b) All t except for $t = n$ for n an odd integer.

(c) For t an odd integer, $f(t) = -1$. Fourier series converges to 0.

6. (a)



(b) The function is continuous for all $t \neq 2n+1$ so the Fourier series converges to $f(t)$ for all $t \neq 2n+1$.

(c) For $t = 2n+1$, $f(t) = f(-1) = -3$. The Fourier series converges to $(-3+1)/2 = -1$.

12. The Fourier series of $f(t)$ is

$$f(t) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos(2k+1)t}{(2k+1)^2}.$$

Since $f(t)$ is continuous, the Fourier series converges to $f(t)$ for all t. Thus,

$$|t| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\cos(2k+1)t}{(2k+1)^2} \quad \text{for } -\pi < t < \pi.$$

Letting $t = 0$ gives

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2},$$

so that

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

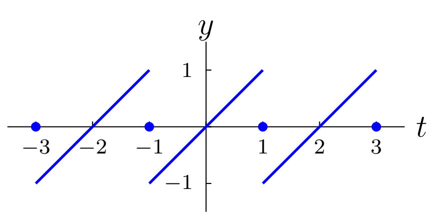
Section 10.4: 2, 8

2. The odd extension of $f(t)$ is the sawtooth wave of period 2 given by $f_o(t) = t$ for $-1 \le t < 1$; $f(t+2) = f(t)$. This is exactly the Fourier series computed in Example 5, Section 10.2, with $L = 1$ so the Fourier sine series of $f(t)$ is

$$t \sim \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin n\pi t.$$

This series converges to the sawtooth wave for all $t \neq n$ where n is an odd integer. At $t = n$ is an odd integer, the series converges to 0.

Graph:



The even extension of $f(t)$ is the even triangular wave function of period 2:

$$f(t) = \begin{cases} -t & \text{if } -1 \le t < 0, \\ t & \text{if } 0 \le t < 1 \end{cases}, f(t+2) = f(t).$$

The Fourier coefficients are:

For $n = 0$,

$$\begin{aligned} a_0 &= 2\int_0^1 f(t)\,dt \\ &= 2\int_0^1 t\,dt = 2\left.\frac{t^2}{2}\right|_0^1 = 1. \end{aligned}$$

For $n \geq 1$,

$$\begin{aligned} a_n &= 2\int_0^1 f(t)\cos n\pi t\,dt \\ &= 2\int_0^1 t\cos n\pi t\,dt \qquad \left(\text{let } x = n\pi t \text{ so } t = \frac{x}{n\pi} \text{ and } dt = \frac{dx}{n\pi}\right) \\ &= 2\int_0^{n\pi} \frac{x}{n\pi}\cos x\,\frac{dx}{n\pi} = \frac{2}{n^2\pi^2}\left[x\sin x + \cos x\right]_{x=0}^{x=n\pi} \\ &= \frac{2}{n^2\pi^2}[\cos n\pi - 1] = \frac{2}{n^2\pi^2}[(-1)^n - 1] \end{aligned}$$

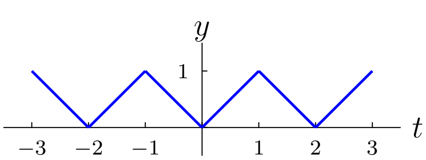
Therefore,

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -\frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

and the Fourier series is

$$\begin{aligned} f(t) &\sim \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{\cos\pi t}{1^2} + \frac{\cos 3\pi t}{3^2} + \frac{\cos 5\pi t}{5^2} + \frac{\cos 7\pi t}{7^2} + \cdots\right) \\ &= \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=\text{odd}}^{\infty}\frac{\cos n\pi t}{n^2}. \end{aligned}$$

Since the even extension is continuous, it converges to the even extension at all points t.

Graph: 

8. The odd extension of $f(t) = \sin t$ defined on $0 < t < \pi$ is just $f_o(t) = \sin t$ for all $t \in \mathbb{R}$. Thus, the Fourier sine series is just $\sin t$.

The even extension of $f(t) = \sin t$ is $f_e(t) = |\sin t|$ for all $t \in \mathbb{R}$. The coefficients of the Fourier cosine series are thus:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\sin t\,dt = -\frac{2}{\pi}\cos t\Big|_0^{\pi} = -\frac{2}{\pi}(\cos\pi - \cos 0) = \frac{4}{\pi},$$

and for $n \geq 1$, (the product formula $\sin\theta\cos\phi = \frac{1}{2}(\sin(\theta+\phi)+\sin(\theta-\phi))$ will be useful),

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi \sin t \cos nt\,dt = \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}(\sin(n+1)t - \sin(n-1)t)\,dt \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)t + \frac{1}{n-1}\cos(n-1)t\right]_0^\pi \\
&= \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)\pi + \frac{1}{n+1} + \frac{1}{n-1}\cos(n-1)\pi - \frac{1}{n-1}\right] \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{2}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] & \text{if } n \text{ is even} \end{cases} \\
&= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{\pi(n^2-1)} & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

Thus, the Fourier series is

$$\begin{aligned}
f(t) &\sim \frac{2}{\pi} + \frac{4}{\pi}\left(-\frac{1}{3}\cos 2t - \frac{1}{15}\cos 4t - \frac{1}{35}\cos 6t + \cdots\right) \\
&= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=\text{even}}^{\infty}\frac{1}{n^2-1}\cos nt.
\end{aligned}$$

Graph: