Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the Fourier series supplement:

Section 10.5: 2 (a)

2 (a) The hypotheses of Theorem 7 (termwise integration) are satisfied since

$$g(\pi) = \int_{-\pi}^{\pi} f_1(t)\,dt = \int_{-\pi}^{\pi} t\,dt = \left.\frac{t^2}{2}\right|_{\pi}^{\pi} = 0.$$

Thus, Theorem 7 applies, and the Fourier series of $g(t) = \int_{\pi}^{t} x\,dx$ is

$$g(t) = \frac{t^2}{2} - \frac{\pi^2}{2} \sim \frac{A_0}{2} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nt,$$

where

$$A_0 = -\frac{1}{\pi}\int_{-\pi}^{\pi} t f_1(t)\,dt = -\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\,dt = -\frac{1}{\pi}\left.\frac{t^3}{3}\right|_{-\pi}^{\pi} = -\frac{2\pi^2}{3}.$$

Thus, solving for t^2 gives the Fourier series

$$f_2(t) = t^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos nt.$$

Section 10.6: 6

6 The Fourier series of $f(t)$ is the sine series of $f(t)$. It was computed in Exercise 2 of Section 10.4 as $f(t) \sim \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi t}{n}$. Let $y(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos n\pi t + B_n \sin n\pi t)$ be a 2-periodic solution of $y'' + 5y = f(t)$ expressed as the sum of its Fourier series. Then $y(t)$ will satisfy the hypotheses of Theorem 3 (termwise differentiation) of Section 10.5. Thus, differentiating twice will give

$$y''(t) = \sum_{n=1}^{\infty}\left[-n^2\pi^2 A_n \cos n\pi t - n^2\pi^2 B_n \sin n\pi t\right].$$

Substituting into the differential equation gives

$$\begin{aligned} y''(t) + y(t) &= \frac{5A_0}{2} + \sum_{n=1}^{\infty}\left[A_n(5 - n^2\pi^2)\cos n\pi t + B_n(5 - n^2\pi^2)\sin n\pi t\right] \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin n\pi t}{n}. \end{aligned}$$

Comparing corresponding coefficients of $\cos n\pi t$ and $\sin n\pi t$ gives the equations

$$\begin{aligned} \frac{5A_0}{2} &= 0 \\ A_n(5 - n^2\pi^2) &= 0 \qquad \text{for all } n \geq 1 \\ B_n(5 - n^2\pi^2) &= \frac{2(-1)^{n+1}}{n\pi} \qquad \text{for all } n \geq 1 \end{aligned}$$

Solving these equations gives $A_0 = 0$ for all $n \geq 0$,

$$B_n = \frac{2(-1)^{n+1}}{(5 - n^2\pi^2)n\pi}.$$

Thus, the unique 2-periodic solution is the sum of the Fourier series expansion

$$y(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(5 - n^2\pi^2)n\pi} \sin n\pi t.$$

Do the following problem from the heat equation supplement:
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9. From the definition of the heat equation (Equation 12.1.4, page 624 of heat equation supplement), $a^2 = 9$, $L = 4$, and $f(x) = 1$ for $0 < x < 4$. Thus, the solution of the heat equation is given from Equation 12.1.5 (page 624) by

$$u(x,\, t) = \sum_{n=1}^{\infty} \alpha_n e^{-9n^2\pi^2 t/16} \sin \frac{n\pi x}{4},$$

where α_n is the n^{th} coefficient of the Fourier Sine series of $f(x)$. Thus,

$$\begin{aligned} \alpha_n &= \frac{2}{4}\int_0^4 \sin \frac{n\pi x}{4}\, dx \\ &= -\frac{2}{n\pi} \cos \frac{n\pi x}{4}\bigg|_0^4 \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Hence, the solution of the heat equation is

$$u(x,\, t) = \frac{4}{\pi} \sum_{n=\text{odd}} \frac{1}{n} e^{-9n^2\pi^2 t/16} \sin \frac{n\pi x}{4}.$$