**Instructions.** Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the text:
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**6.**

$$BC = \begin{bmatrix} 2 & 1 & 1 & -3 \\ 0 & -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 3 & 1 \\ 0 & 1 & 8 \\ 1 & 1 & 7 \end{bmatrix}$$
$$= \begin{bmatrix} 2{\cdot}2+1{\cdot}1+1{\cdot}0+(-3){\cdot}1 & 2{\cdot}1+1{\cdot}3+1{\cdot}1+(-3){\cdot}1 & 2{\cdot}2+1{\cdot}1+1{\cdot}8+(-3){\cdot}7 \\ 0{\cdot}2+(-1){\cdot}1+4{\cdot}0+(-1){\cdot}1 & 0{\cdot}1+(-1){\cdot}3+4{\cdot}1+(-1){\cdot}1 & 0{\cdot}2+(-1){\cdot}1+4{\cdot}8+(-1){\cdot}7 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 3 & -8 \\ -2 & 0 & 24 \end{bmatrix}$$

**18.** $AB = \begin{bmatrix} 1 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 4 \cdot 0 + 3 \cdot (-1) + 1 \cdot (-2) \end{bmatrix} = \begin{bmatrix} -4 \end{bmatrix}$

$$BA = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 & 1 \cdot 4 & 1 \cdot 3 & 1 \cdot 1 \\ 0 \cdot 1 & 0 \cdot 4 & 0 \cdot 3 & 0 \cdot 1 \\ -1 \cdot 1 & -1 \cdot 4 & -1 \cdot 3 & -1 \cdot 1 \\ -2 \cdot 1 & -2 \cdot 4 & -2 \cdot 3 & -2 \cdot 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 4 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & -3 & -1 \\ -2 & -8 & -6 & -2 \end{bmatrix}$$

Solve the following systems of linear equations.

1. $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & 5x_2 & + & 8x_3 & = & 2 \\ 5x_1 & + & 8x_2 & + & 11x_3 & = & 14 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 8 & 2 \\ 5 & 8 & 11 & 14 \end{bmatrix}.$$

Now row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 5 & 8 & 2 \\ 5 & 8 & 11 & 14 \end{bmatrix} \xrightarrow[t_{13}(-5)]{t_{12}(-2)} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & -2 & -4 & 4 \end{bmatrix}$$

$$\xrightarrow{t_{23}(-2)} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{t_{21}(-2)} \begin{bmatrix} 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row echelon form. It is the augmented matrix of the linear system

$$\begin{array}{ccccccc} x_1 & + & & - & x_3 & = & 6 \\ & & x_2 & + & 2x_3 & = & -2 \end{array}$$

where the last equation $0 = 0$ is left off. The solutions of this last system are obtained by identifying the bound variables as $x_1$ and $x_2$, so there is one free variable $x_3$. Solving for $x_1$ and $x_2$ in terms of $x_3$ gives:

$$x_1 = 6 + x_3$$
$$x_2 = -2 + 2x_3.$$

Letting the free variable be assigned arbitrarily: $x_3 = \alpha$ gives all the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6+\alpha \\ 2+2\alpha \\ \alpha \end{bmatrix}$$

where $\alpha$ is an arbitrary real number. ◀

2. $$\begin{array}{ccccccc} 2x_1 & + & x_2 & - & x_3 & = & 3 \\ & & x_2 & + & 2x_3 & = & 1 \\ 4x_1 & + & 2x_2 & - & 2x_3 & = & 7 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 4 & 2 & -2 & 7 \end{bmatrix}.$$

Now row reduce the augmented matrix:

$$\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 4 & 2 & -2 & 7 \end{bmatrix} \xrightarrow{t_{13}(-2)} \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The third row of this last matrix corresponds to the equation $0 = 1$, which has no solutions. Thus, the original system of equations also has no solutions. ◀

3. $$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & - & x_3 & = & 1 \\ x_1 & - & 2x_2 & + & 6x_3 & = & 11 \\ 5x_1 & + & x_2 & - & x_3 & = & 12 \end{array}$$

▶ **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -2 & 6 & 11 \\ 5 & 1 & -1 & 12 \end{bmatrix}.$$

Now row reduce the augmented matrix:

$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -2 & 6 & 11 \\ 5 & 1 & -1 & 12 \end{bmatrix} \xrightarrow{p_{1\,2}} \begin{bmatrix} 1 & -2 & 6 & 11 \\ 2 & 3 & -1 & 1 \\ 5 & 1 & -1 & 12 \end{bmatrix}$$

$$\xrightarrow[t_{1\,3}(-5)]{t_{1\,2}(-2)} \begin{bmatrix} 1 & -2 & 6 & 11 \\ 0 & 7 & -13 & -21 \\ 0 & 11 & -31 & -43 \end{bmatrix} \xrightarrow{m_1(1/7)} \begin{bmatrix} 1 & -2 & 6 & 11 \\ 0 & 1 & -13/7 & -3 \\ 0 & 11 & -31 & -43 \end{bmatrix}$$

$$\xrightarrow{t_{2\,3}(-11)} \begin{bmatrix} 1 & -2 & 6 & 11 \\ 0 & 1 & -13/7 & -3 \\ 0 & 0 & -74/7 & -10 \end{bmatrix} \xrightarrow{m_3(-7/74)} \begin{bmatrix} 1 & -2 & 6 & 11 \\ 0 & 1 & -13/7 & -3 \\ 0 & 0 & 1 & 35/37 \end{bmatrix}$$

$$\xrightarrow[t_{3,\,1}(-6)]{t_{3\,2}(13/7)} \begin{bmatrix} 1 & -2 & 0 & 197/37 \\ 0 & 1 & 0 & -46/37 \\ 0 & 0 & 1 & 35/37 \end{bmatrix} \xrightarrow{t_{2\,1}(2)} \begin{bmatrix} 1 & 0 & 0 & 105/37 \\ 0 & 1 & 0 & -46/37 \\ 0 & 0 & 1 & 35/37 \end{bmatrix}$$

The last matrix is in reduced row echelon form. It is the augmented matrix of the linear system

$$\begin{array}{ccccr} x_1 & & & = & 105/37 \\ & x_2 & & = & -46/37 \\ & & x_3 & = & 35/37 \end{array}$$

Since all variables are bound, there is a unique solution

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 105/37 \\ -46/37 \\ 35/37 \end{bmatrix}.$$

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4. $$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 3x_3 & + & 2x_4 & = & 2 \\ 2x_1 & + & 5x_2 & - & 8x_3 & + & 6x_4 & = & 5 \\ 3x_1 & + & 4x_2 & - & 5x_3 & + & 2x_4 & = & 4 \end{array}$$

► **Solution.** The augmented matrix of the system is

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{bmatrix}.$$

Now row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & 2 & 2 \\ 2 & 5 & -8 & 6 & 5 \\ 3 & 4 & -5 & 2 & 4 \end{bmatrix} \underset{t_{1\,3}(-3)}{\overset{t_{1\,2}(-2)}{\longrightarrow}} \begin{bmatrix} 1 & 2 & -3 & 2 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & -2 & 4 & -4 & -2 \end{bmatrix}$$

$$\overset{t_{2\,3}(2)}{\longrightarrow} \begin{bmatrix} 1 & 2 & -3 & 2 & 2 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \overset{t_{2\,1}(-2)}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 & 2- & 0 \\ 0 & 1 & -2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in reduced row echelon form. It is the augmented matrix of the linear system

$$\begin{array}{ccccccccc} x_1 & + & & & x_3 & - & 2x_4 & = & 0 \\ & & x_2 & - & 2x_3 & + & 2x_4 & = & 1 \end{array}$$

where the last equation $0 = 0$ is left off. The solutions of this last system are obtained by identifying the bound variables as $x_1$ and $x_2$, so the free variables are $x_3$ and $x_4$. Solving for $x_1$ and $x_2$ in terms of $x_3$ and $x_4$, gives:

$$\begin{aligned} x_1 &= -x_3 + 2x_4 \\ x_2 &= 1 + 2x_3 - 2x_4. \end{aligned}$$

Letting the free variables be assigned arbitrarily: $x_3 = \alpha$, $x_4 = \beta$ gives all the solutions as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\alpha + 2\beta \\ 1 + 2\alpha - 2\beta \\ \alpha \\ \beta \end{bmatrix}$$

where $\alpha$ and $\beta$ are arbitrary real numbers. ◄