Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the text:
Page 602: 6, 10,16
Page 615: 12, 20
Section 8.3:

6. Form the matrix

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Now row reduce this matrix:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[t_{31}(-1)]{t_{32}(-2)} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{t_{21}(-1)} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

10. Form the matrix

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now row reduce this matrix:

$$\begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{t_{13}(1) \\ t_{14}(-1)}]{t_{12}(1)} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & -2 & -2 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{t_{34}(1)} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{t_{42}(1)} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & -2 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{t_{23}(-1)} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow[p_{34}]{p_{23}} \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\substack{m_3(-\frac{1}{2}) \\ m_4(-\frac{1}{2})}]{\substack{m_1(-1) \\ m_2(\frac{1}{2})}} \begin{bmatrix} 1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{t_{41}(-1)} \begin{bmatrix} 1 & -1 & -1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\xrightarrow{t_{31}(1)} \begin{bmatrix} 1 & -1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \xrightarrow{t_{21}(1)} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{2} \begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix}.$$

16. Form the matrix

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 5 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

Now row reduce this matrix:

$$\begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ -2 & 5 & -2 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{t_{12}(2)} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 0 & 2 & 1 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{m_2(1/3)} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/3 & 1/3 & 0 \\ 0 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{t_{23}(-2)} \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & -1 & -4/3 & -2/3 & 1 \end{bmatrix}$$

$$\xrightarrow[m_3(-1)]{t_{31}(1)} \begin{bmatrix} 1 & -1 & 0 & -1/3 & -2/3 & 1 \\ 0 & 1 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 4/3 & 2/3 & -1 \end{bmatrix} \xrightarrow{t_{21}(1)} \begin{bmatrix} 1 & 0 & 0 & 1/3 & -1/3 & 1 \\ 0 & 1 & 0 & 2/3 & 1/3 & 0 \\ 0 & 0 & 1 & 4/3 & 2/3 & -1 \end{bmatrix}$$

Therefore,

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & -3 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{3}\begin{bmatrix} 1 & -1 & 3 \\ 2 & 1 & 0 \\ 4 & 2 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Section 8.4:

12. $sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} s-1 & -1 \\ 1 & s-1 \end{bmatrix}$ so $\det(sI - A) = (s-1)^2 + 1 = s^2 - 2s + 2$. Thus, $\det A = 0$ for $s = 1 \pm i$, and for $s \neq 1 \pm i$,

$$(sI - A)^{-1} = \frac{1}{s^2 - 2s + 2}\begin{bmatrix} s-1 & 1 \\ -1 & s-1 \end{bmatrix}.$$

20. For $A = \begin{bmatrix} 4 & 0 & 3 \\ 8 & 1 & 7 \\ 3 & 4 & 1 \end{bmatrix}$, the cofactor matrix is

$$\text{Cofac(A)} = \begin{bmatrix} \begin{vmatrix} 1 & 7 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 8 & 7 \\ 3 & 1 \end{vmatrix} & \begin{vmatrix} 8 & 1 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 3 \\ 4 & 1 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 3 & 1 \end{vmatrix} & -\begin{vmatrix} 4 & 0 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 0 & 3 \\ 1 & 7 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 8 & 7 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 8 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -27 & 13 & 29 \\ 12 & -5 & -16 \\ -3 & -4 & 4 \end{bmatrix}.$$

Then $\det A = 4(-27) - 0(13) + 3(29) = -21$ and

$$\text{Adj}(A) = \text{Cofac(A)}^t = \begin{bmatrix} -27 & 12 & -3 \\ 13 & -5 & -4 \\ 29 & -16 & 4 \end{bmatrix},$$

so

$$A^{-1} = \frac{1}{\det(A)}\,\text{Adj}(A) = \frac{1}{21}\begin{bmatrix} 27 & -12 & 3 \\ -13 & 5 & 4 \\ -29 & 16 & -4 \end{bmatrix}.$$

Additional problems:

1. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 3 & -1 & 2 \end{bmatrix}$, if possible.

▶ **Solution.** Form the matrix

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 3 & -1 & 2 & 0 & 0 & 1 \end{bmatrix}.$$

Now row reduce this matrix:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[t_{13}(-3)]{t_{12}(-1)} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[t_{23}(-1)]{m_2(-1)} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -2 & -1 & 1 \end{bmatrix}$$

Since the first 3 entries of the third row are 0, A^{-1} does not exist. ◀

2. Compute the inverse of the matrix $A = \begin{bmatrix} e^t \sin 2t & -e^{-t} \cos 2t \\ e^t \cos 2t & e^{-t} \sin 2t \end{bmatrix}$, if possible.

▶ **Solution.** Use the adjoint formula. First,

$$\det A = (e^t \sin 2t)(e^{-t} \sin 2t) - (-e^{-t} \cos 2t)(e^t \cos 2t) = \sin^2 2t + \cos^2 2t = 1.$$

Then

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} e^{-t} \sin 2t & e^{-t} \cos 2t \\ -e^t \cos 2t & e^t \sin 2t \end{bmatrix} = \begin{bmatrix} e^{-t} \sin 2t & e^{-t} \cos 2t \\ -e^t \cos 2t & e^t \sin 2t \end{bmatrix}.$$

◀

3. Compute the determinant of the following matrices:

$$(a)\ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 4 & 0 & 0 \\ 0 & 5 & -1 & 0 \\ 11 & 0 & -2 & 2 \end{bmatrix} \qquad (b) \begin{bmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -1 & 5 \\ -1 & -2 & 1 & 7 \\ 1 & 1 & 0 & -6 \end{bmatrix}$$

▶ **Solution.** (a) Since the matrix A is lower triangular, the determinant of A is the product of the diagonal entries. That is $\det A = 1 \cdot 4 \cdot (-1) \cdot 2 = -8$

(b) By adding the first row of the matrix to the third row and subtracting the first row from the fourth row, the determinant is not changed. Thus,

$$\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -1 & 5 \\ -1 & -2 & 1 & 7 \\ 1 & 1 & 0 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -1 & 5 \\ 0 & 1 & 1 & 5 \\ 0 & -2 & 0 & -4 \end{vmatrix}.$$

Now using cofactor expansion along the first column gives

$$\begin{vmatrix} 1 & 3 & 0 & -2 \\ 0 & 1 & -1 & 5 \\ 0 & 1 & 1 & 5 \\ 0 & -2 & 0 & -4 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 5 \\ 1 & 1 & 5 \\ -2 & 0 & -4 \end{vmatrix}.$$

Compute this last determinant by cofactor expansion along the second column:

$$\begin{aligned}\begin{vmatrix} 1 & -1 & 5 \\ 1 & 1 & 5 \\ -2 & 0 & -4 \end{vmatrix} &= (-1)(-1)^{1+2}\begin{vmatrix} 1 & 5 \\ -2 & -4 \end{vmatrix} + 1\cdot(-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -2 & -4 \end{vmatrix} \\ &= 2\begin{vmatrix} 1 & 5 \\ -2 & -4 \end{vmatrix} \\ &= 2(-4-(-2)\cdot 5) = 12.\end{aligned}$$

Thus, the determinant of the original matrix is 12. ◀

4. Solve the following system by Cramer's rule. Here λ is an unspecified number.

$$\begin{aligned} x + 2y &= \lambda \\ 3x + 2y &= 1 \end{aligned}$$

► **Solution.** From Cramer's rule,

$$x = \frac{\begin{vmatrix} \lambda & 2 \\ 1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{2\lambda - 2}{2-6} = \frac{1-\lambda}{2},$$

and

$$y = \frac{\begin{vmatrix} 1 & \lambda \\ 3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{1-3\lambda}{2-6} = \frac{3\lambda - 1}{4}.$$

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