Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the text:
Section 1.1: 12, 14, 20
Section 1.3: 10, 14, 24, 36
Section 1.4: 4, 8, 12, 18, 28
Section 1.1:

12. The following table summarizes the needed calculations:

Function	$y'(t)$	$2y(t)$
$y_1(t) = 0$	$y_1'(t) = 0$	$2y_1(t) = 0$
$y_2(t) = t^2$	$y_2'(t) = 2t$	$2y_2(t) = 2t^2$
$y_3(t) = 3e^{2t}$	$y_3'(t) = 6e^{2t}$	$2y_3(t) = 6e^{2t}$
$y_4(t) = 2e^{3t}$	$y_4'(t) = 6e^{3t}$	$2y_4(t) = 4e^{3t}$

To be a solution, the entries in the second and third columns need to be the same. Thus y_1 and y_3 are the only solutions.

14. We first write the differential equation in standard form: $y'' = -4y$. The following table summarizes the needed calculations:

Function	$y''(t)$	$-4y(t)$
$y_1(t) = e^{2t}$	$y_1''(t) = 4e^{2t}$	$-4y_1(t) = -4e^{2t}$
$y_2(t) = \sin 2t$	$y_2''(t) = -4\sin 2t$	$-4y_2(t) = -4\sin 2t$
$y_3(t) = \cos(2t-1)$	$y_3''(t) - 4\cos(2t-1)$	$-4y_3(t) = -4\cos(2t-1)$
$y_4(t) = t^2$	$y_4''(t) = 2$	$-4y_4(t) = -4t^2$

To be a solution, the entries in the second and third columns need to be the same. Thus y_2 and y_3 are solutions.

20.

$$y'(t) = -ce^{-t} + 3$$
$$-y(t) + 3t = -ce^{-t} - 3t + 3 + 3t = -ce^{-t} + 3.$$

Note that $y(t)$ is defined for all $t \in \mathbb{R}$.

Section 1.3:

10. The variables are already separated, so integrate both sides of the equation to get $y^2/2 = t^2/2 + c$, which we can rewrite as $y^2 - t^2 = k$ where $k = 2c \in \mathbb{R}$ is a constant. Since $y(2) = -1$, substitute $t = 2$ and $y = -1$ to get that $k = (-1)^2 - 2^2 = -3$. Thus the solution is given implicitly by the equation $y^2 - t^2 = -3$ or we can solve explicitly to get $y = -\sqrt{t^2 - 3}$, where the negative square root is used since $y(2) = -1 < 0$.

14. There is an equilibrium solution $y = 0$. Separating variables give $y^{-2}y' = t$ and integrating gives $-y^{-1} = t^2/2 + c$. Thus $y = -2/(t^2 + 2c)$, c a real constant. This is equivalent to writing $y = -2/(t^2 + c)$, c a real constant, since twice an arbitrary constant is still an arbitrary constant.

24. In standard form we get $y' = y(y + 1)$ from which we see $y = 0$ and $y = -1$ are equilibrium solutions. The equilibrium solution $y(t) = 0$ satisfies the initial condition $y(0) = 0$ so $y(t) = 0$ is the required solution.

36. The ambient temperature is $T_a = 70°$. Equation (13), page 35, gives $T(t) = 70 + ke^{rt}$ for the temperature of the coffee at time t. Since the initial temperature of the coffee is $T(0) = 180$ we get $180 = T(0) = 70 + k$. Thus $k = 110$. The constant r is determined from the temperature at a second time: $140 = T(3) = 70 + 110e^{3r}$ so $r = \frac{1}{3}\ln\frac{7}{11} \approx -.1507$. Thus $T(t) = 70 + 110e^{rt}$, with r as calculated. The temperature requested is $T(5) = 70 + 110\left(\frac{7}{11}\right)^{\frac{5}{3}} \approx 121.8°$.

Section 1.4:

4. Divide by t to put the equation in the standard form

$$y' + \frac{1}{t}y = \frac{e^t}{t}$$

In this case $p(t) = 1/t$, an antiderivative is $P(t) = \ln t$, and the integrating factor is $\mu(t) = t$. Now multiply the standard form equation by the integrating factor to get $ty' + y = e^t$, the left hand side of which is a perfect derivative $(ty)'$. (Note that this is just the original left hand side of the equation. Thus if we had recognized that the left hand side was already a perfect derivative, the preliminary steps could have been skipped for this problem, and we could have proceeded directly to the next step.) Thus the equation can be written as $(ty)' = e^t$ and taking antiderivatives of both sides gives $ty = e^t + c$ where $c \in \mathbb{R}$ is a constant. Now divide by t to get

$$y = \frac{e^t}{t} + \frac{c}{t}$$

for the general solution.

8. In this case $p(t) = 2$ and the integrating factor is $e^{\int 2\,dt} = e^{2t}$. Now multiply to get $e^{2t}y' + 2e^{2t}y = e^{2t}\sin t$, which simplifies to $(e^{2t}y)' = e^{2t}\sin t$. Now integrate both sides to get $e^{2t}y = \dfrac{1}{5}(-\cos t + 2\sin t)e^{2t} + c$, where we computed $\int e^{2t}\sin t$ by parts two times. Dividing by e^{2t} gives $y = \dfrac{1}{5}(2\sin t - \cos t) + ce^{-2t}$.

12. The given differential equation is in standard form, $p(t) = a$, an antiderivative is $P(t) = at$, and the integrating factor is $\mu(t) = e^{at}$. Now multiply by the integrating factor to get

$$e^{at}y' + ae^{at}y = be^{at},$$

the left hand side of which is a perfect derivative $((e^{at})y)'$. Thus

$$((e^{at})y)' = be^{at}.$$

If $a \neq 0$ then taking antiderivatives of both sides gives $e^{at}y = \frac{b}{a}e^{at} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by e^{-at} to get $y = \frac{b}{a} + ce^{-at}$ for the general solution. In the case $a = 0$ then $y' = b$ and $y = bt + c$.

18. The given equation is in standard form, $p(t) = a$, an antiderivative is $P(t) = at$, and the integrating factor is $\mu(t) = e^{at}$. Now multiply by the integrating factor to get $e^{at}y' + ae^{at}y = t^n$, the left hand side of which is a perfect derivative $(e^{at}y)'$. Thus $(e^{at}y)' = t^n$ and taking antiderivatives of both sides gives $e^{at}y = \frac{t^{n+1}}{n+1} + c$ where $c \in \mathbb{R}$ is a constant. Now multiply by e^{-at} to get $y = \frac{t^{n+1}}{n+1}e^{-at} + ce^{-at}$ for the general solution.

28. Let $y(t)$ be the amount of salt in the tank at time t. Since pure water is initially in the tank, the initial value is $y(0) = 0$. Determine a differential equation for $y(t)$ from the input and output rate of salt:

input rate: input rate $= 1\ \dfrac{\text{L}}{\text{min}} \times 10\ \dfrac{\text{g}}{\text{L}} = 10\ \dfrac{\text{g}}{\text{min}}$.

output rate: output rate $= 1\ \dfrac{\text{L}}{\text{min}} \times \dfrac{y(t)}{10}\ \dfrac{\text{g}}{\text{L}} = \dfrac{y(t)}{10}\ \dfrac{\text{g}}{\text{min}}$.

Since $y' =$ input rate$-$output rate, it follows that $y(t)$ satisfies the initial value problem

$$y' = 10 - \frac{1}{10}y\,, \quad y(0) = 0.$$

Put in standard form, this equation becomes $y' + \frac{1}{10}y = 10$. The coefficient function is $p(t) = \frac{1}{10}$, $P(t) = \int p(t)\,dt = t/10$, and the integrating factor is $\mu(t) = e^{t/10}$. Multiplying the standard form equation by the integrating factor gives $(e^{t/10}y)' = 10e^{t/10}$. Integrating and simplifying gives $y = 100 + ce^{-t/10}$. The initial condition $y(0) = 0$ implies $c = -100$ so $y = 100 - 100e^{-t/10}$. After 10 minutes we have $y(10) = 100 - 100e^{-1}$ g of salt. The concentration is thus $(100 - 100e^{-1})/10 = 10 - 10e^{-1}$ g/L