Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the text:
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12. The mass is $m = 32/32 = 1$ slug. The spring constant k is given by $k = 32/2 = 16$. Since no external force is mentioned we may assume it is zero. The initial conditions are $y(0) = -1$ and $y'(0) = 0$. The equation $y'' + 8y' + 16y = 0$, $y(0) = -1$, $y'(0) = 0$ models the motion of the body. The characteristic polynomial is $q(s) = (s+4)^2$. Thus

$$y = c_1 e^{-4t} + c_2 t e^{-4t}.$$

The initial conditions imply $c_1 = 1$ and $c_2 = 4$. Thus

$$y = (1 + 4t)e^{-4t}.$$

The discriminant of the characteristic equation is $D = 8^2 - 4 \cdot 1 \cdot 16 = 0$ so the motion is critically damped free motion. The equation $y(t) = 0$ implies $t = -1/4$. So there is no positive time where the body crosses equilibrium.

14. The mass is $m = 2$. The spring constant k is given by $k = 5/1 = 5$. The damping constant is given by $\mu = 6/1 = 6$. Since no external force is mentioned we may assume it is zero. The initial conditions are $y(0) = .1$ and $y'(0) = .1$. The equation $2y'' + 6y' + 5y = 0$, $y(0) = .1$, $y'(0) = .1$ models the motion of the body. The characteristic polynomial is $q(s) = 2s^2 + 6s + 5 = 2\left(\left(s + \frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)$. Thus

$$y = c_1 e^{-\frac{3}{2}t} \cos \frac{1}{2}t + c_2 e^{-\frac{3}{2}t} \sin \frac{1}{2}t.$$

The initial conditions imply $c_1 = \frac{1}{10}$ and $c_2 = \frac{1}{2}$. So

$$y = \frac{1}{10} e^{-\frac{3}{2}t} \cos \frac{1}{2}t + \frac{1}{2} e^{-\frac{3}{2}t} \sin \frac{1}{2}t.$$

The discriminant of the characteristic equation is $D = 6^2 - 4 \cdot 2 \cdot 5 = -4 < 0$ so the motion is underdamped free motion and crosses equilibrium. If $A = \left(\frac{1}{100} + \frac{1}{4}\right)^{\frac{1}{2}} = \frac{\sqrt{26}}{10}$ and $\phi = \arctan 5 = 1.3734$ then we can write $y = \frac{\sqrt{26}}{10} e^{-\frac{3}{2}t} \cos\left(\frac{1}{10}t - 1.3734\right)$.

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12.

1. $y(t) = 1$
2. $y(t) = t$
3. $y(t) = e^{-t}$
4. $y(t) = \cos 2t$

18. If $y = \frac{1}{6}t^5$ then $y' = \frac{5}{6}t^4$ and $y'' = \frac{20}{6}t^3$ so that $\mathbf{L}y = t^2\left(\frac{20}{6}t^3\right) - 4t\left(\frac{5}{6}t^4\right) + 6\left(\frac{1}{6}t^5\right) = t^5$. Part (1) follows. If $y = t^2$ then $\mathbf{L}y = t^2(2) - 4t(2t) + 6(t^2) = 0$. It follows that $y = t^2$ is a solution to $\mathbf{L}y = 0$. If $y = t^3$ then $\mathbf{L}y = t^2(6t) - 4t(3t^2) + 6t^3 = 0$. Part (2) now follows. By linearity every function of the form $y(t) = \frac{1}{6}t^5 + c_1t^2 + c_2t^3$ is a solution to $\mathbf{L}y = t^5$, where c_1 and c_2 are constants. If we want a solution to $\mathbf{L}(y) = t^5$ with $y(0) = a$ and $y'(0) = b$, then we need to solve for c_1 and c_2: Since $y(t) = \frac{1}{6}t^5c_1t^2 + c_2t^3$ we have $y'(t) = \frac{5}{6}t^5 + 2c_1t + 3c_2t^2$. Hence,

$$\begin{aligned} a &= y(1) = \frac{1}{6} + c_1 + c_2 \\ b &= y'(1) = \frac{5}{6} + 2c_1 + 3c_2. \end{aligned}$$

These equations give $c_1 = 3a - b + \frac{1}{3}$ and $c_2 = b - 2a - \frac{1}{2}$. Particular choices of a and b give the answers for Part (3).

(3)a. $y = \frac{1}{6}t^5 + \frac{10}{3}t^2 - \frac{5}{2}t^3$

(3)b. $y = \frac{1}{6}t^5 - \frac{2}{3}t^2 + \frac{1}{2}t^3$

(3)c. $y = \frac{1}{6}t^5 - \frac{17}{3}t^2 + \frac{9}{2}t^3$

(3)d. $y = \frac{1}{6}t^5 + \left(\frac{1}{3} + 3a - b\right)t^2 + \left(-\frac{1}{2} - 2a + b\right)t^3$

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2. The indicial polynomial is $Q(s) = 2s^2 - 7s + 3 = (2s-1)(s-3)$. There are two distinct roots $\frac{1}{2}$ and 3. The fundamental set is $\left\{t^{\frac{1}{2}}, t^3\right\}$. The general solution is $y(t) = c_1t^{\frac{1}{2}} + c_2t^3$.

8. The indicial polynomial is

$$Q(s) = s^2 - s + 1 = s^2 - s + \frac{1}{4} + \frac{3}{4} = \left(s - \frac{1}{2}\right)^2 + \quad \frac{\sqrt{3}}{2}\Bigg)^2.$$

There are two complex roots, $\frac{1+i\sqrt{3}}{2}$ and $\frac{1-i\sqrt{3}}{2}$. The fundamental set is

$$\left\{t^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\ln t\right), t^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\ln t\right)\right\},$$

and the general solution is $y(t) = c_1t^{\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}\ln t\right) + c_2t^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\ln t\right)$.

10. The indicial polynomial is $Q(s) = s^2 + 4$. There are two complex roots, $2i$ and $-2i$. The fundamental set is $\{\cos(2\ln t), \sin(2\ln t)\}$. The general solution is $y(t) = c_1\cos(2\ln t) + c_2\sin(2\ln t)$.

14. The indicial polynomial is $Q(s) = s^2 + 4$. There are two complex roots, $\pm 2i$. The fundamental set is $\{\cos(2\ln t), \sin(2\ln t)\}$. The general solution is $y(t) = c_1\cos(2\ln t) + c_2\sin(2\ln t)$. The initial conditions imply

$$\begin{aligned} c_1 &= -3 \\ 2c_2 &= 4. \end{aligned}$$

Thus $c_1 = -3$ and $c_2 = 2$. Hence $y = -3\cos(2\ln t) + 2\sin(2\ln t)$

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8. A fundamental set is $\{\cos t, \sin t\}$. The matrix equation $\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \sec t \end{pmatrix}$ implies $u_1'(t) = -\tan t$ and $u_2'(t) = 1$. Hence $u_1(t) = \ln(\cos t)$, $u_2(t) = t$, and $y_p(t) = \cos t \ln(\cos t) + t\sin t$. The general solution is $y(t) = \cos t\ln(\cos t) + t\sin t + c_1\cos t + c_2\sin t$.

14. When put in standard form one sees that $f(t) = te^{-t}$. The matrix equation $\begin{pmatrix} t-1 & e^{-t} \\ 1 & -e^{-t} \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ te^{-t} \end{pmatrix}$ implies $u_1'(t) = e^{-t}$ and $u_2'(t) = 1 - t$. Hence $u_1(t) = -e^{-t}$, $u_2(t) = t - \frac{t^2}{2}$, and $y_p(t) = -(t-1)e^{-t} + (t - \frac{t^2}{2})e^{-t} = \frac{-t^2}{2}e^{-t} + e^{-t}$. Since e^{-t} is a homogeneous solution we can write the general solution as $y(t) = \frac{-t^2}{2}e^{-t} + c_1(t-1) + c_2e^{-t}$.