Instructions. Do each problem showing your work. Answers alone are not sufficient. Label each problem clearly, and write neatly, in a logical sequence.

Do the following problems from the text:
Page 645: 8

8. First note that $y_1(0) = 1$ and $y_2(0) = 0$, so the initial condition is satisfied. Then

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} e^t + 2e^t + 2te^t \\ 4e^t + 4te^t \end{bmatrix} = \begin{bmatrix} 3e^t + 2te^t \\ 4e^t + 4te^t \end{bmatrix}$$

while

$$\begin{aligned} \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} e^t + 2te^t \\ 4te^t \end{bmatrix} &= \begin{bmatrix} 3e^t + 6te^t - 4te^t \\ 4e^t + 8te^t - 4te^t \end{bmatrix} \\ &= \begin{bmatrix} 3e^t + 2te^t \\ 4e^t + 4te^t \end{bmatrix}. \end{aligned}$$

Thus $\mathbf{y}'(t) = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \mathbf{y}(t)$, as required.

Page 655: 8, 10, 12

8. The characteristic polynomial is $c_A(s) = (s-1)(s-2)$ and $sI - A = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}$. Thus $(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}$ and

$$e^{At} = \mathcal{L}^{-1}(sI - A)^{-1} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

10. The characteristic polynomial is $c_A(s) = (s-1)^2 - 1 = s(s-2)$ and $sI - A = \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}$. Thus the resolvent matrix is $(sI - A)^{-1} = \begin{bmatrix} \frac{s-1}{s(s-2)} & \frac{1}{s(s-2)} \\ \frac{1}{s(s-2)} & \frac{s-1}{s(s-2)} \end{bmatrix}$ A partial fraction decomposition gives

$$(sI - A)^{-1} = \frac{1}{s} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + \frac{1}{s-2} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

and therefore

$$e^{At} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} + e^{2t} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} & -\frac{1}{2} + \frac{1}{2}e^{2t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} & \frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

12. The characteristic polynomial is $c_A(s) = s^2 - 2s + 2 = (s-1)^2 + 1$ and $sI - A = \begin{bmatrix} s-4 & 10 \\ -1 & s+2 \end{bmatrix}$. Thus the resolvent matrix is $(sI - A)^{-1} = \begin{bmatrix} \frac{s+2}{(s-1)^2+1} & \frac{-10}{(s-1)^2+1} \\ \frac{1}{(s-1)^2+1} & \frac{s-4}{(s-1)^2+1} \end{bmatrix}$ which we write as

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s-1}{(s-1)^2+1} & -\frac{10}{(s-1)^2+1} \\ \frac{1}{(s-1)^2+1} & \frac{s-1}{(s-1)^2+1} \end{bmatrix} + \begin{bmatrix} \frac{3}{(s-1)^2+1} & 0 \\ 0 & -\frac{3}{(s-1)^2+1} \end{bmatrix}.$$

Therefore

$$e^{At} = \begin{bmatrix} e^t \cos t + 3e^t \sin t & -10e^t \sin t \\ e^t \sin t & e^t \cos t - 3e^t \sin t \end{bmatrix}$$

Page 663: 4, 6, 14

4. The characteristic polynomial is $s^2 - 5s + 6 = (s-2)(s-3)$ and has roots $2, 3$. The standard basis of $\mathcal{E}_{c_A}$ is $\mathcal{B}_{c_A} = \{e^{2t},\, e^{3t}\}$. It follows that

$$e^{At} = Me^{2t} + Ne^{3t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M + N \\ A &= 2M + 3N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= 3I - A \\ N &= A - 2I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= (3I - A)e^{2t} + (A - 2I)e^{3t} \\ &= \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} e^{3t} \\ &= \begin{bmatrix} 2e^{2t} - e^{3t} & -e^{2t} + e^{3t} \\ 2e^{2t} - 2e^{3t} & -e^{2t} + 2e^{3t} \end{bmatrix} \end{aligned}$$

6. The characteristic polynomial is $c_A(s) = (s-3)^2$. The standard basis is $\mathcal{B}_{c_A} = \{e^{3t},\, te^{3t}\}$. It follows that

$$e^{At} = Me^{3t} + Nte^{3t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M \\ A &= 3M + N. \end{aligned}$$

from which we get

$$\begin{aligned} M &= I \\ N &= A - 3I. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Ie^{3t} + (A - 3I)te^{3t} \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix} + \begin{bmatrix} te^{3t} & -te^{3t} \\ te^{3t} & -te^{3t} \end{bmatrix} \\ &= \begin{bmatrix} e^{3t} + te^{3t} & -te^{3t} \\ te^{3t} & e^{3t} - te^{3t} \end{bmatrix} \end{aligned}$$

14. In this case $\mathcal{B}_{c_A} = \{e^t,\, te^t,\, e^{2t}\}$. It follows that

$$e^{At} = Me^t + Nte^t + Pe^{2t}.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} M + P &= I \\ M + N + 2P &= A \\ M + 2N + 4P &= A^2 \end{aligned}$$

from which we get

$$\begin{aligned} M &= 2A - A^2 \\ N &= -A^2 + 3A - 2I \\ P &= I - 2A + A^2. \end{aligned}$$

Thus,

$$\begin{aligned} e^{At} &= Me^t + Nte^t + Pe^{2t} \\ &= \begin{bmatrix} -2 & 1 & 2 \\ 0 & 1 & 0 \\ -3 & 1 & 3 \end{bmatrix} e^t + \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix} te^t + \begin{bmatrix} 3 & -1 & -2 \\ 0 & 0 & 0 \\ 3 & -1 & -2 \end{bmatrix} e^{2t} \\ &= \begin{bmatrix} -2e^t - te^t + 3e^{2t} & e^t - e^{2t} & 2e^t + te^t - 2e^{2t} \\ -te^t & e^t & te^t \\ -3e^t - te^t + 3e^{2t} & e^t - e^{2t} & 3e^t + te^t - 2e^{2t} \end{bmatrix} \end{aligned}$$

Page 677: 4, 6, 8, 10

4. The characteristic polynomial is $c_A(s) = (s+1)^2+4$. Thus $\mathcal{B}_{c_A} = \{e^{-t}\cos 2t, e^{-t}\sin 2t\}$ and

$$e^{At} = M_1e^{-t}\cos 2t + M_2e^{-t}\sin 2t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= -M_1 + 2M_2. \end{aligned}$$

and hence $M_1 = I$ and $M_2 = \frac{1}{2}(A + I)$. We thus get $e^{At} = \begin{bmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix}$ and $\mathbf{y}(t) = \begin{bmatrix} e^{-t}\cos 2t & e^{-t}\sin 2t \\ -e^{-t}\sin 2t & e^{-t}\cos 2t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e^{-t}\cos 2t \\ -e^{-t}\sin 2t \end{bmatrix}$

6. The characteristic polynomial is $c_A(s) = s^2 + 1$. Thus $\mathcal{B}_{c_A} = \{\cos t, \sin t\}$ and

$$e^{At} = M_1 \cos t + M_2 \sin t.$$

Differentiating and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 \\ A &= M_2. \end{aligned}$$

We thus get

$$e^{At} = \cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin t \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$$

and hence

$$\begin{aligned} \mathbf{y}_h(t) &= e^{At}\mathbf{y}_0 \\ &= \left(\cos t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin t \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix} \end{aligned}$$

8. The characteristic polynomial is $c_A(s) = (s+1)(s-2)(s-1)$. Thus $\mathcal{B}_{c_A} = \{e^{-t}, e^{2t}, e^{t}\}$ and

$$e^{At} = M_1 e^{-t} + M_2 e^{2t} + M_3 e^{t}.$$

Differentiating twice and evaluating at $t = 0$ gives

$$\begin{aligned} I &= M_1 + M_2 + M_3 \\ A &= -M_1 + 2M_2 + M_3 \\ A^2 &= M_1 + 4M_2 + M_3. \end{aligned}$$

Solving this system gives

$$\begin{aligned} M_1 &= \frac{1}{6}(A^2 - 3A + 2I) = \begin{bmatrix} 1 & 0 & -3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M_2 &= \frac{1}{3}(A^2 - I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ M_3 &= -\frac{1}{2}(A^2 - A - 2I) = \begin{bmatrix} 0 & 0 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

We thus get $e^{At} = \begin{bmatrix} e^{-t} & 0 & \frac{3}{2}(e^t - e^{-t}) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix}$ and

$$\mathbf{y}(t) = \begin{bmatrix} e^{-t} & 0 & \frac{3}{2}(e^t - e^{-t}) \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2e^{-t} + 3e^t \\ e^{2t} \\ 2e^t \end{bmatrix}.$$

10. A straightforward calculation gives

$$e^{At} = \frac{e^t}{2} \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} + \frac{e^{-t}}{2} \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}.$$

It follows that

$$\begin{aligned} \mathbf{y}_h(t) &= e^{At}\mathbf{y}_0 \\ &= \frac{e^{-t}}{2} \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \frac{e^{-t}}{2} \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\ &= e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}_p &= \int_0^t e^{A(t-u)}\mathbf{f}(u)\,du \\ &= \int_0^t \frac{e^{(t-u)}}{2} \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} + \frac{e^{-(t-u)}}{2} \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \Bigg) \begin{bmatrix} e^u \\ e^u \end{bmatrix} du \\ &= \int_0^t \frac{e^t}{2} \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{e^{-t+2u}}{2} \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Bigg) du \\ &= te^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned}$$

It now follows that

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{y}_h + \mathbf{y}_p \\ &= \begin{bmatrix} e^{-t} + te^t \\ 3e^{-t} + te^t \end{bmatrix} \end{aligned}$$