Math 2070 Section 1
Review Exercises for Exam 1
Answers

1. ► **Solution.** $AB = \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 8+x & 16 \\ 8 & 6 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 & x \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 14 & x+10 \\ 8 & x \end{bmatrix}$. These matrices are equal if $8+x = 14$, $16 = x+10$, $8 = 8$, and $6 = x$. These are all satisfied for $\boxed{x = 6}$. ◄

2. $\boxed{(BA)^T}$

3. ► **Solution.** Use Gauss-Jordan elimination on the augmented matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 1 & 3 & 0 & 1 & 7 \\ 1 & 0 & 2 & 1 & 3 \end{bmatrix} \xrightarrow[R_3 \mapsto R_3 - R_1]{R_2 \mapsto R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & -2 & -1 & 0 & -5 \end{bmatrix}$$

$$\xrightarrow[R_3 \mapsto R_3 + 2R_2]{} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & -7 & 0 & -7 \end{bmatrix} \xrightarrow{-\frac{1}{7}R_2} \begin{bmatrix} 1 & 2 & 3 & 1 & 8 \\ 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[R_1 \mapsto R_1 - 3R_3]{R_2 \mapsto R_2 + 3R_3} \begin{bmatrix} 1 & 2 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

The last matrix is in reduced row-echelon form and is the augmented matrix of the linear system

$$\begin{array}{ccccccccc} x_1 & & & & & + & x_4 & = & 1 \\ & & x_2 & & & & & = & 2 \\ & & & & x_3 & & & = & 1 \end{array}$$

The solution set of this last system is the same as that of the original system and is given by:

$$\mathcal{S} = \{(1 - x_4,\, 2,\, 1,\, x_4) : \; x_4 \text{ is arbitrary}\}.$$

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4. (a) ▶ **Solution.** Use Gauss-Jordan elimination of the augmented matrix $\left[A \mid I_3\right]$:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - 2R_1} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 2 & 1 & -2 & 1 & 0 \end{bmatrix} \xrightarrow{R_3 \mapsto R_3 - 2R_2} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$\xrightarrow{R_1 \mapsto R_1 + R_2} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Hence,

$$A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

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(b) ▶ **Solution.**

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

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5. ▶ **Solution.** Start by trying to row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 2 & -3 & b_1 \\ 2 & 3 & 3 & b_2 \\ 5 & 9 & -6 & b_3 \end{bmatrix} \begin{matrix} R_2 \mapsto R_2 - 2R_1 \\ \longrightarrow \\ R_3 \mapsto R_3 - 5R_1 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & -1 & 9 & b_2 - 2b_1 \\ 0 & -1 & 9 & b_3 - 5b_1 \end{bmatrix}$$

$$\begin{matrix} -R_2 \\ \longrightarrow \\ R_3 \mapsto R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & 2 & -3 & b_1 \\ 0 & 1 & 9 & -b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 - 3b_1 - b_2 \end{bmatrix}$$

In order for the system to be consistent, the last entry in the last column must be 0, and this entry being 0 is also sufficient to be able to complete the row reduction and solve the system. Hence, the system is consistent if and only if $\boxed{3b_1 + b_2 = b_3}$. ◀

6. ▶ **Solution.** (a) Use cofactor expansion along the second column to get

$$|A| = (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} + k\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - k^2\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 + 2k^2.$$

   (b) A is not invertible if and only if $\det A = 0$ which is true if and only if $k^2 - 1 = 0 \iff k = \pm 1$.

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7. (a) L
   (b) S
   (c) S
   (d) N
   (e) S, L
   (f) L

8. (a) $y(t) = ce^{-2t}$
   (b) $y(t) = ce^{-2t} + e^t$
   (c) $y(t) = ce^t + \frac{1}{2}e^{3t}$
   (d) $y(t) = ce^{-t^2} + \frac{1}{2}$
   (e) $y(t) = ct^{-3} - t^{-3}\cos t$
   (f) $y^2 - t^2 = c$
   (g) $y(t) = 2\tan(2t + c)$

9. (a) $y(t) = 5te^{2t} - e^{2t}$
   (b) $y(t) = -4/(t^4 + 3)$
   (c) $y(t) = -2e^{-3t}\cos 2t + e^{-3t}$
   (d) $y(t) = t^4 - \frac{2}{t^3}$

10. $T' = k(T - 350)$, $T(0) = 70$ (k is a proportionality constant)

11. (a) $y(t) = 600 - 550e^{-0.01t}$ (b) 600 lbs. This makes sense because, after a long period of time, the amount of salt in the tank should be close to the "steady state" value which corresponds to the concentration of the solution flowing into the tank. That is, 300 gal times 2 lb/gal = 600 lbs of salt.