Math 2070 Section 1
Review Exercises for Exam II

The syllabus for Exam II is Sections 2.1 – 2.7 of Chapter 2, Sections 3.1 – 3.4 of Chapter 3, and Sections 4.1 – 4.3 of Chapter 4. You should review all of the assigned exercises in these sections. Note that Section 2.2 contains the main formulas for computation of Laplace transforms. A Laplace transform table will be provided with the test. Following is a brief list of terms, skills, and formulas with which you should be familiar.

- Know how to use all of the Laplace transform formulas developed in Section 2.2 to be able to compute the Laplace transform of elementary functions.

- Know how to use partial fraction decompositions to be able to compute the inverse Laplace transform of any proper rational function. The key recursion algorithms for computing partial fraction decompositions are Theorem 1 (Page 129) for the case of a real root in the denominator, and Theorem 1 (Page 143) for a complex root in the denominator. Here are the two results:

  Theorem 1 (Linear Partial Fraction Recursion). *Let $P_0(s)$ and $Q(s)$ be polynomials. Assume that a is a number such that $Q(a) \neq 0$ and n is a positive integer. Then there is a unique number A_1 and polynomial $P_1(s)$ such that*

  $$\frac{P_0(s)}{(s-a)^n Q(s)} = \frac{A_1}{(s-a)^n} + \frac{P_1(s)}{(s-a)^{n-1}Q(s)}.$$

  The number A_1 and polynomial $P_1(s)$ are computed as follows:

  $$A_1 = \left.\frac{P_0(s)}{Q(s)}\right|_{s=a} \quad \text{and} \quad P_1(s) = \frac{P_0(s) - A_1 Q(s)}{s-a}.$$

  Theorem 2 (Quadratic Partial Fraction Recursion). *Let $P_0(s)$ and $Q(s)$ be polynomials. Assume that $a+bi$ is a complex number with nonzero imaginary part $b \neq 0$ such that $Q(a+bi) \neq 0$ and n is a positive integer. Then there is a unique linear term $B_1 s + C_1$ and polynomial $P_1(s)$ such that*

  $$\frac{P_0(s)}{((s-a)^2+b^2)^n Q(s)} = \frac{B_1 s + C_1}{((s-a)^2+b^2)^n} + \frac{P_1(s)}{((s-a)^2+b^2)^{n-1}Q(s)}.$$

  The linear term $B_1 s + C_1$ and polynomial $P_1(s)$ are computed as follows:

  $$B_1 s + C_1|_{s=a+bi} = \left.\frac{P_0(s)}{Q(s)}\right|_{s=a+bi} \quad \text{and} \quad P_1(s) = \frac{P_0(s) - (B_1 s + C_1)Q(s)}{(s-a)^2+b^2}.$$

- Know how to apply the input derivative principle (Corollary 8, Page 116) to compute the solution of an initial value problem for a constant coefficient linear differential equation with elementary forcing function. See Section 2.1.

- Know how to use the *characteristic polynomial* to be able to solve constant coefficient homogeneous linear differential equations. (See Algorithm 6, Page 232, and Algorithm 3, Page 286.)

- Know how to use the *method of undetermined coefficients* to find a particular solution of the constant coefficient linear differential equation

$$q(\boldsymbol{D}) = f(t)$$

where the forcing function $f(t)$ is an exponential polynomial, i.e., $f(t)$ is a sum of functions of the form $ct^k e^{at} \cos bt$ and $dt^k e^{at} \sin bt$ for various choices of the constants a, b, c, and d, and non-negative integer k. (See Algorithm 4, Page 239 and Algorithm 3, Page 295.)

The following is a small set of exercises of types identical to those already assigned.

1. (a) Complete the following definition: Suppose $f(t)$ is a continuous function of exponential type defined for all $t \geq 0$. The **Laplace transform** of f is the function $F(s)$ defined as follows

$$F(s) = \mathcal{L}(f(t))(s) = \boxed{\int_0^\infty e^{-st} f(t)\, dt}$$

for all s sufficiently large.

(b) Using your definition compute the Laplace transform of the function $f(t) = 2t - 5$. You may need to review the integration by parts formula: $\int u\, dv = uv - \int v\, du$.

▶ **Solution.** The Laplace transform of $f(t) = 2t - 5$ is the integral

$$\mathcal{L}(2t-5)(s) = \int_0^\infty (2t-5)e^{-st}\, dt,$$

which is computed using the integration by parts formula by letting $u = 2t - 5$ and $dv = e^{-wt}\, dt$, so that $du = 2\, dt$ while $v = -\frac{1}{s}e^{-st}$. Thus, if $s > 0$,

$$\begin{aligned}
\mathcal{L}(2t-5)(s) &= \int_0^\infty (2t-5)e^{-st}\, dt \\
&= -\frac{2t-5}{s}e^{-st}\Big|_0^\infty + \int_0^\infty \frac{2}{s}e^{-st}\, dt \\
&= \left(-\frac{2t-5}{s}e^{-st} - \frac{2}{s^2}e^{-st}\right)\Big|_0^\infty \\
&= -\frac{5}{s} + \frac{2}{s^2}.
\end{aligned}$$

The last evaluation uses the fact (verified in calculus) that $\lim_{t\to\infty} e^{-st} = 0$ and $\lim_{t\to\infty} te^{-st} = 0$ provided $s > 0$. ◀

2. Compute the Laplace transform of each of the following functions using the Laplace transform Tables. (A table of Laplace Transforms will be provided to you on the exam.)

(a) $3t^3 - 2t^2 + 7$

$$\boxed{3\frac{3!}{s^4} - 2\frac{2!}{s^3} + \frac{7}{s} = \frac{18}{s^4} - \frac{4}{s^3} + \frac{7}{s}.}$$

(b) $e^{-3t} + \sin\sqrt{2}t$

$$\boxed{\frac{1}{s+3} + \frac{\sqrt{2}}{s^2+2}.}$$

(c) $-8 + \cos(t/2)$

$$\boxed{-\frac{8}{s} + \frac{2}{s^2+1/4} = -\frac{8}{s} + \frac{4s}{4s^2+1}.}$$

(d) $7e^{2t}\cos 3t - 2e^{7t}\sin 5t$

$$\boxed{\frac{7s}{(s-2)^2+9} - \frac{10}{(s-7)^2+25}.}$$

(e) $(2-t^2)e^{-5t}$

▶ **Solution.** Use the first translation principle. Then

$$\boxed{\frac{2}{s+5} - \frac{2}{(s+5)^3}.}$$

◀

(f) $t\cos 6t$

$$\boxed{-\frac{d}{ds}\left(\frac{s}{s^2+36}\right) = \frac{s^2-36}{(s^2+36)^2}}$$

(g) $t^2\cos at$ where a is a constant

▶ **Solution.** Use the transform derivative principal twice, applied to $f(t) = \cos at$. Then, $F(s) = s/(s^2+a^2)$ and $\mathcal{L}\left\{t^2\cos at\right\}(s) = F''(s)$. Since $F'(s) = (a^2-s^2)/(s^2+a^2)^2$, the Laplace transform of $t^2\cos at$ is

$$\boxed{F''(s) = \frac{2s^2 - 6sa^2}{(s^2+a^2)^3}.}$$

◀

3. Find the inverse Laplace transform of each of the following functions. You may use the Laplace Transform Tables.

(a) $\dfrac{7}{(s+3)^3}$ $\boxed{\dfrac{7}{2}t^2e^{-3t}.}$

(b) $\dfrac{s+2}{s^2-3s-4}$

▶ **Solution.** Use partial fractions to write

$$G(s) = \frac{s+2}{s^2-3s-4} = \frac{1}{5}\left(\frac{6}{s-4} - \frac{1}{s+1}\right).$$

Thus $\boxed{\left(6e^{4t} - e^{-t}\right)/5}$. ◀

(c) $\dfrac{s}{(s+4)^2+4}$

▶ **Solution.** Since

$$\frac{s}{(s+4)^2+4} = \frac{(s+4)-4}{(s+4)^2+4} = \frac{s+4}{(s+4)^2+4} - 2\frac{2}{(s+4)^2+4},$$

it follows that the inverse Laplace transform $\boxed{e^{-4t}\cos 2t - 2e^{-4t}\sin 2t}$. ◀

(d) $\dfrac{1}{s^2-10s+9}$

▶ **Solution.** Since $s^2-10s-9 = (s-9)(s-1)$, use partial fractions:

$$\frac{1}{s^2-10s+9} = \frac{1}{8}\left(\frac{1}{s-9} - \frac{1}{s-1}\right) \Rightarrow \boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2-10s+9}\right\} = \frac{1}{8}(e^{9t} - e^t).}$$

◀

(e) $\dfrac{2s+18}{s^2+25}$ $\boxed{2\cos 5t + (18/5)\sin 5t}$

(f) $\dfrac{4s-3}{s^2-6s+25}$

▶ **Solution.** Since $s^2-6s+25 = (s-3)^2+4^2$, replace s in the numerator by $(s-3)+3$ to get

$$\frac{4s-3}{s^2-6s+25} = \frac{4(s-3)+9}{(s-3)^2+4^2} = \frac{4(s-3)}{(s-3)^2+4^2} + \frac{9}{(s-3)^2+4^2}.$$

We conclude:

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s-3}{s^2-6s+25}\right\} = 4e^{3t}\cos 4t + \frac{9}{4}e^{3t}\sin 4t.}$$

◀

(g) $\dfrac{1}{s(s^2+4)}$

► **Solution.** Use partial fractions to write

$$\frac{1}{s(s^2+4)} = \frac{1}{4}\left(\frac{1}{s} - \frac{s}{s^2+4}\right),$$

so that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}\left(1-\cos 2t\right).}$$

◄

(h) $\dfrac{1}{s^2(s+1)^2}$

► **Solution.** Use partial fractions to write

$$\frac{1}{s^2(s+1)^2} = \frac{1}{s^2} - \frac{2}{s} + \frac{1}{(s+1)^2} + \frac{2}{s+1},$$

so that

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\} = te^{-t} + 2e^{-t} + t - 2.}$$

◄

4. Solve each of the following differential equations by means of the Laplace transform:

(a) $y'' - 3y' + 2y = 4, \qquad y(0) = 2, \quad y'(0) = 3$

► **Solution.** As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation gives

$$s^2Y(s) - 2s - 3 - 3(Y(s) - 2) + 2Y(s) = \frac{4}{s}$$

and solving for $Y(s)$ gives:

$$\begin{aligned} Y(s) &= \frac{2s^2 - 3s + 4}{s(s-2)(s-1)} \\ &= \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \\ &= \frac{2}{s} - \frac{3}{s-1} + \frac{3}{s-2}, \end{aligned}$$

where the last two lines represent the decomposition of $Y(s)$ into partial fractions. Taking the inverse Laplace transform gives

$$\boxed{y(t) = 2 - 3e^t + 3e^{2t}.}$$

◄

(b) $y'' + 4y = 6\sin t, \qquad y(0) = 6, \quad y'(0) = 0$

▶ **Solution.** As usual, $Y = \mathcal{L}(y)$. Applying $\mathcal{L}$ to both sides of the equation and solving for Y gives:

$$Y(s) = \frac{6s}{s^2+4} + \frac{6}{(s^2+4)(s^2+1)} = \frac{6s}{s^2+4} + \frac{2}{s^2+1} - \frac{2}{s^2+4}.$$

Taking the inverse Laplace transform gives

$$\boxed{y(t) = 6\cos 2t + 2\sin t - \sin 2t.}$$

◀

5. Using the Laplace transform, find the solution of the following differential equations with initial conditions $y(0) = 0$, $y'(0) = 0$:

   (a) $y'' - y = 2\sin t$ $\quad\boxed{y(t) = (1/2)(e^t - e^{-t}) - \sin t}$

   (b) $y'' + 2y' = 5y$ $\quad\boxed{y(t) = 0}$

   (c) $y'' + y = \sin 4t$ $\quad\boxed{y(t) = (1/15)(4\sin t - \sin 4t)}$

   (d) $y'' + y' = 1 + 2t$ $\quad\boxed{y(t) = 1 - e^{-t} + t^2 - t}$

   (e) $y'' + 4y' + 3y = 6$ $\quad\boxed{y(t) = e^{-3t} - 3e^{-t} + 2}$

   (f) $y'' - 2y' = 3(t + e^{2t})$ $\quad\boxed{y(t) = (3/8)(1 - 2t - 2t^2 - e^{2t} + 4te^{2t}}$

   (g) $y'' - 2y' = 20e^{-t}\cos t$ $\quad\boxed{y(t) = 3e^{2t} - 5 + 2e^{-t}(\cos t - 2\sin t)}$

   (h) $y'' + y = 2 + 2\cos t$ $\quad\boxed{y(t) = 2 - 2\cos t + t\sin t}$

   (i) $y'' - y' = 30\cos 3t$ $\quad\boxed{y(t) = 3e^t - 3\cos 3t - \sin 3t}$

6. Solve each of the following homogeneous linear differential equations, using the techniques of Chapter 3 (Characteristic equation and calculation of the standard basis $\mathcal{B}_q$.)

   (a) $y'' + 3y' + 2y = 0$

   (b) $y'' + 6y' + 13y = 0$

   (c) $y'' + 6y' + 9y = 0$

   (d) $y'' - 2y' - y = 0$

   (e) $8y'' + 4y' + y = 0$

   (f) $2y'' - 7y' + 5y = 0$

(g) $2y'' + 2y' + y = 0$

(h) $y'' + .2y' + .01y = 0$

(i) $y'' + 7y' + 12y = 0$

(j) $y'' + 2y' + 2y = 0$

(k) $y''' + 2y'' - 15y' = 0$

(l) $y''' + 2y'' - 8y' = 0$

(m) $y''' - 2y'' - 3y' = 0$

(n) $y''' - 7y' + 6y = 0$

(o) $y''' - 3y'' - y' + 3y = 0$

(p) $4y''' - 10y' + 12y = 0$

(q) $y^{(4)} - 5y'' + 4y = 0$

Answers

(a) $y = c_1e^{-t} + c_2e^{-2t}$

(b) $y = c_1e^{-3t}\cos 2t + c_2e^{-3t}\sin 2t$

(c) $y = (c_1 + c_2t)e^{-3t}$

(d) $y = c_1e^{(1+\sqrt{2})t} + c_2e^{(1-\sqrt{2})t}$

(e) $y = e^{-t/4}(c_1\cos\frac{t}{4} + c_2\sin\frac{t}{4})$

(f) $y = c_1e^{5t/2} + c_2e^{t}$

(g) $y = e^{-t/2}(c_1\cos(-t/2) + c_2\sin(t/2))$

(h) $y = e^{0.1t}(c_1 + tc_2)$

(i) $y = c_1e^{-4t} + c_2e^{-3t}$

(j) $y = e^{-t}(c_1\cos t + c_2\sin t)$

(k) $y = c_1 + c_2e^{3t} + c_3e^{-5t}$

(l) $y = c_1 + c_2e^{2t} + c_3e^{-4t}$

(m) $y = c_1 + c_2e^{-t} + c_3e^{3t}$

(n) $y = c_2e^{t} + c_2e^{2t} + c_3e^{-3t}$

(o) $y = c_1e^{3t} + c_2e^{t} + c_3e^{-t}$

(p) $y = c_1e^{-2t} + c_2e^{t/2} + c_3e^{3t/2}$

(q) $y = c_1e^{2t} + c_2e^{-2t} + c_3e^{t} + c_4e^{-t}$

7. Find the general solution of the constant coefficient homogeneous linear differential equation with the given characteristic polynomial $q(s)$.

   (a) $q(s) = (s-1)(s+3)(s-5)$

   (b) $q(s) = s^3 - 1$

(c) $q(s) = (s^2 - 2)^2$

(d) $q(s) = s^3 - 3s^2 + s + 5$

(e) $q(s) = s^4 + 3s^2 - 4$

(f) $q(s) = s^4 + 5s^2 + 4$

(g) $q(s) = (s^2 + 1)^3$

(h) $q(s)$ has degree 4 and has roots $\sqrt{2}$ with multiplicity 2 and $2 \pm 3i$ with multiplicity 1.

(i) $q(s)$ has degree 5 and roots 0 with multiplicity 3 and $1 \pm \sqrt{3}$ with multiplicity 1.

Answers

(a) Roots of $q(s)$ are 4, -3, 5, so $y(t) = c_1 e^t + c_2 e^{-3t} + c_3 e^{5t}$.

(b) $s^3 - 1 = (s-1)(s^2+s+1)$ so the roots are 1, $(-1 \pm \sqrt{3}i)/2$. Thus $y(t) = c_1 e^t + c_2 e^{-t/2} \cos \sqrt{3}/2 + c_3 e^{-t/2} \sin \sqrt{3}/2$.

(c) Roots are $\pm\sqrt{2}$, each of multiplicity 2. Thus, $y(t) = e^{\sqrt{2}t}(c_1 + c_2 t) + e^{-\sqrt{2}t}(c_3 + c_4 t)$.

(d) -1 is a root since $q(-1) = 0$, so $q(s) = (s+1)(s^2 - 4s + 5)$ and the roots are -1 and $2 \pm i$. Thus, $y(t) = c_1 e^{-t} + e^{2t}(c_2 \cos t + c_3 \sin t)$.

(e) $q(s) = (s^2 + 4)(s^2 - 1)$ so the roots are ± 1 and $\pm 2i$. Thus, $y(t) = c_1 e^t + c_2 e^{-t} + c_3 \cos 2t + c_4 \sin 2t$.

(f) $q(s) = (s^2 + 4)(s^2 + 1)$ so the roots are $\pm i$ and $\pm 2i$. Thus, $y(t) = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t$.

(g) The roots are $\pm i$, each with multiplicity 3. Thus, $y(t) = (c_1 + c_2 t + c_3 t^2) \cos t + (c_4 + c_5 t + c_6 t^2) \sin t$.

(h) $y(t) = (c_1 + c_2 t) e^{\sqrt{2}t} + e^{2t}(c_3 \cos t + c_4 \sin t)$

(i) $y(t) = (c_1 + c_2 t + c_3 t^2) + e^t(c_4 \cos t + c_5 \sin t)$

8. Solve each of the following initial value problems. You may (and should) use the work already done in exercise 6.

(a) $y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = -3.$

(b) $y'' + 6y' + 13y = 0, \quad y(0) = 0, \quad y'(0) = -1.$

(c) $y'' + 6y' + 9y = 0, \quad y(0) = -1, \quad y'(0) = 5.$

(d) $y'' - 2y' - y = 0, \quad y(0) = 0, \quad y'(0) = \sqrt{2}.$

(e) $y'' + 2y' + 2y = 0, \quad y(0) = 0, \quad y'(0) = 2$

Answers

(a) $y = 2e^{-2t} - e^{-t}$

(b) $y = -\frac{1}{2} e^{-3t} \sin 2t$

(c) $y = (-1 + 2t)e^{-3t}$

(d) $y = \frac{1}{2}(e^{(1+\sqrt{2})t} - e^{(1-\sqrt{2})t})$

(e) $y = 2e^{-t}\sin t$

9. Use the method of undetermined coefficients (See Section 3.4) to find the general solution of each of the following differential equations.

(a) $y'' - 3y' - 4y = 30e^t$

(b) $y'' - 3y' - 4y = 30e^{4t}$

(c) $y'' - 3y' - 4y = 20\cos t$

(d) $y'' - 2y' + y = t^2 - 1$

(e) $y'' - 2y' + y = 3e^{2t}$

(f) $y'' - 2y' + y = 4\cos t$

(g) $y'' - 2y' + y = 3e^t$

(h) $y'' - 2y' + y = te^t$

Answers

(a) $y = c_1e^{4t} + c_2e^{-t} - 5e^t$

(b) $y = c_1e^{4t} + c_2e^{-t} + 6te^{4t}$

(c) $y = c_1e^{4t} + c_2e^{-t} + (-30/17)\sin t + (-150/51)\cos t$

(d) $y = c_1e^t + c_2te^t + t^2 + 4t + 5$

(e) $y = c_1e^t + c_2te^t + 3e^{2t}$

(f) $y = c_1e^t + c_2te^t + -2\sin t$

(g) $y = c_1e^t + c_2te^t + \frac{3}{2}t^2e^t$

(h) $y = c_e^t + c_2te^t + \frac{1}{6}t^3e^t$