The syllabus for Exam 3 is Sections 3.6, 5.1 to 5.3, 5.6, and 6.1 to 6.4. You should review the assigned exercises in these sections. Following is a brief list (not necessarily complete) of terms, skills, and formulas with which you should be familiar.

- Know how to set up the equation of motion of a spring system with dashpot:

  $$my'' + \mu y' + ky = f(t)$$

  where m is the mass, μ is the damping constant, k is the spring constant, and $f(t)$ is the applied force. Know how to determine if the system is *underdamped, critically damped, or overdamped.*

- Know how to solve the *Cauchy-Euler equation*

  $$at^2y'' + bty' + cy = 0$$

  where a, b, and c are real constants by means of the roots r_1, r_2 of the indicial equation

  $$q(r) = ar(r-1) + br + c = 0.$$

- If $\{y_1(t),\, y_2(t)\}$ is a fundamental solution set for the homogeneous equation

  $$y'' + b(t)y' + c(t)y = 0,$$

  know how to use the method of *variation of parameters* to find a particular solution y_p of the non-homogeneous equation

  $$y'' + b(t)y' + c(t)y = f(t).$$

  In this method, it is not necessary for $f(t)$ to be an elementary function.

  Variation of Parameters: Find y_p in the form $y_p = u_1y_1 + u_2y_2$ where u_1 and u_2 are unknown functions whose derivatives satisfy the following two equations:

  $$(*) \qquad \begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= f(t). \end{aligned}$$

  Solve the system $(*)$ for u_1' and u_2', and then integrate to find u_1 and u_2.

- Know what it means for a function to have a *jump discontinuity* and to be *piecewise continuous.*

- Know how to piece together solutions on different intervals to produce a solution of one of the initial value problems

  $$y' + ay = f(t), \quad y(t_0) = y_0,$$

  or

  $$y'' + ay' + by = f(t), \quad y(t_0) = y_0,\ y'(t_0) = y_1,$$

  where $f(t)$ is a piecewise continuous function on an interval containing t_0.

- Know what the *unit step function* (also called the *Heaviside function*) ($h(t-c)$) and the *on-off switches* ($\chi_{[a,\,b)}$) are:

$$h(t-c) = h_c(t) = \begin{cases} 0 & \text{if } 0 \le t < c, \\ 1 & \text{if } c \ge t \end{cases} \qquad \text{and,}$$

$$\chi_{[a,\,b)} = \begin{cases} 1 & \text{if } a \le t < b, \\ 0 & \text{otherwise} \end{cases} = h(t-a) - h(t-b),$$

  and know how to use these two functions to rewrite a piecewise continuous function in a manner which is convenient for computation of Laplace transforms.

- Know the second translation principle:

$$\mathcal{L}\left\{f(t-c)h(t-c)\right\} = e^{-cs}F(s)$$

  and how to use it (particulary in the form):

$$\mathcal{L}\left\{g(t)h(t-c)\right\} = e^{-cs}\mathcal{L}\left\{g(t+c)\right\}$$

  as a tool for calculating the Laplace transform of piecewise continuous functions.

- Know how to use the second translation principle to compute the inverse Laplace transform of functions of the form $e^{-cs}F(s)$.

- Know how to use the skills in the previous two items in conjunction with the formula for the Laplace transform of a derivative to solve differential equations of the form

$$y' + ay = f(t), \quad y(0) = y_0$$

  and

$$y'' + ay' + by = f(t), \quad y(0) = y_0,\ y'(0) = y_1,$$

  where $f(t)$ is a piecewise continuous forcing function.

- Know the formula for the Laplace transform of the Dirac delta function $\delta(t-c)$:

$$\mathcal{L}\left\{\delta(t-c)\right\} = e^{-cs}.$$

  Know how to use this formula to solve differential equations with impulsive forcing function:

$$y' + ay = K\delta(t-c), \quad y(0) = y_0$$

  and

$$y'' + ay' + by = K\delta(t-c), \quad y(0) = y_0,\ y'(0) = y_1.$$

The following is a small set of exercises of types identical to those already assigned.

1. Solve each of the following Cauchy-Euler differential equations

   (a) $t^2y'' - 7ty' + 15y = 0$

   (b) $4t^2y'' - 8ty' + 9y = 0$

(c) $t^2y'' - 12y = 0$

(d) $t^2y'' + ty' + 16y = 0$

(e) $4t^2y'' - 7ty' + 6y = 0$

(f) $t^2y'' + 5ty' + 4y = 0$

2. Find a solution of each of the following differential equations by using the method of variation of parameters. In each case, $\mathcal{S}$ denotes a fundamental set of solutions of the associated homogeneous equation.

(a) $y'' + 2y' + y = t^{-1}e^{-t}$; $\quad \mathcal{S} = \{e^{-t},\, te^{-t}\}$

(b) $y'' + 9y = 9\sec 3t$; $\quad \mathcal{S} = \{\cos 3t,\, \sin 3t\}$

(c) $t^2y'' + 2ty' - 6y = t^2$; $\quad \mathcal{S} = \{t^2,\, t^{-3}\}$

(d) $y'' + 4y' + 4y = t^{1/2}e^{-2t}$; $\quad \mathcal{S} = \{e^{-2t},\, te^{-2t}\}$

3. Graph each of the following piecewise continuous functions.

(a) $f(t) = \begin{cases} 2 & \text{if } 0 \le t < 1, \\ 3 - 2t & \text{if } t \ge 1. \end{cases}$

(b) $f(t) = e^{-2(t-1)}h(t-1)$

(c) $f(t) = t\chi_{[0,2)} - 3\chi[2,4) + (t-4)^2h(t-4)$

4. Compute the Laplace transform of each of the following functions.

(a) $f(t) = t^2h(t-2)$

(b) $f(t) = (t-1)^2\chi_{[2,4)}$

(c) $f(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ t^2 + t - 1 & \text{if } 1 \le t < 2 \\ -2 & \text{if } t \ge 2. \end{cases}$

5. Compute the inverse Laplace transform of each of the following functions.

(a) $F(s) = \dfrac{se^{-2s}}{s^2 - 9}$

(b) $F(s) = (1 + e^{-\pi s/2})\dfrac{s-1}{s^2 + 2s + 10}$

6. Solve each of the following differential equations.

(a) $y' + 5y = \begin{cases} 2 & \text{if } 0 \le t < 2 \\ t + 1 & \text{if } t \ge 2 \end{cases} \quad y(0) = 0.$

(b) $y'' + 4y = (t+1)h(t-2), \quad y(0) = 0,\, y'(0) = 1.$

(c) $y'' + 4y = 4\delta(t-2), \quad y(0) = -1,\, y'(0) = 1.$

7. A mass hung from a spring behaves according to the equation

$$my'' + \mu y' + ky = 0,$$

where m is the mass, μ is the damping constant, and k is the spring constant. Suppose that $m = 1$ kg, $\mu = p$ kg m, and $k = 2$ N/m (N=Newton). Suppose the following initial conditions are imposed: $y(0) = 1$ m, and $y'(0) = -1$ m/s^2. Determine the values of p for which the system is (a) overdamped, (b) critically damped, (c) underdamped.

8. Find the unique solution of the initial value problem

$$y'' + 2y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1.$$

Is the equation under damped or over damped? Does $y(t) = 0$ for some $t > 0$? If so, find the first $t > 0$ for which $y(t) = 0$. Sketch the graph of your solution.

Answers

1. (a) $y = c_1 t^3 + c_2 t^5$
   (b) $y = t^{3/2}(c_1 + c_2 \ln|t|)$
   (c) $y = c_1 t^4 + c_2 t^{-3}$
   (d) $y = c_1 \cos(4\ln|t|) + c_2 \sin(4\ln|t|)$
   (e) $y = c_1 t^{3/4} + c_2 t^2$
   (f) $y = t^{-2}(c_1 + c_2 \ln|t|)$
2. (a) $y_p(t) = (t\ln|t|)e^{-t}$
   (b) $y_p(t) = (\ln|\cos 3t|)\cos 3t + 3t\sin 3t$
   (c) $y_p(t) = (1/5)t^2 \ln|t|$
   (d) $y_p(t) = (4/15)t^{5/2}e^{-2t}$
4. (a) $F(s) = e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right)$
   (b) $F(s) = e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{2}{s^2} + \dfrac{1}{s}\right) - e^{-4s}\left(\dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}\right)$
   (c) $F(s) = \dfrac{1}{s^2} + e^{-s}\left(\dfrac{2}{s^3} + \dfrac{2}{s^2}\right) - e^{-2s}\left(\dfrac{2}{s^3} + \dfrac{5}{s^2} + \dfrac{7}{s}\right)$
5. (a) $f(t) = \frac{1}{2}\left(e^{3(t-2)} + e^{-3(t-2)}\right)h(t-2)$
   (b) $f(t) = e^{-t}\left(\cos 3t - \frac{2}{3}\sin 3t\right) + e^{-(t-\pi/2)}\left(\cos 3(t-\pi/2) - \frac{2}{3}\sin 3(t-\pi/2)\right)h(t-\pi/2)$
6. (a) $y(t) = \frac{2}{5}(1 - e^{-5t}) + \frac{1}{25}\left(5(t-2) + 4 - 4e^{-5(t-2)}\right)h(t-2)$
   (b) $y(t) = \frac{1}{2}\sin 2t + h(t-2)\left((t-2) + 3 - 3\cos 2(t-2) - \frac{1}{2}\sin 2(t-2)\right)$
   (c) $y(t) = \frac{1}{2}\sin 2t - \cos 2t + 2h(t-2)\sin 2(t-2)$
7. (a) $p > \sqrt{8}$, (b) $p = \sqrt{8}$, (c) $0 < p < \sqrt{8}$
8. ▶ **Solution.** The characteristic polynomial is $p(s) = s^2 + 2s + 5 = (s+1)^2 + 4$ which has roots $-1 \pm 2i$. Thus, the general solution of the homogeneous equation is

$$y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t.$$

We need to choose c_1 and c_2 to satisfy the initial conditions. Since

$$y' = -c_1 e^{-t}\cos 2t - 2c_1 e^{-t}\sin 2t + 2c_2 e^{-t}\cos 2t - c_2 e^{-t}\sin 2t,$$

this means that c_1 and c_2 must satisfy the system of equations

$$\begin{aligned} 3 &= y(0) = c_1 \\ -1 &= y'(0) = -c_1 + 2c_2. \end{aligned}$$

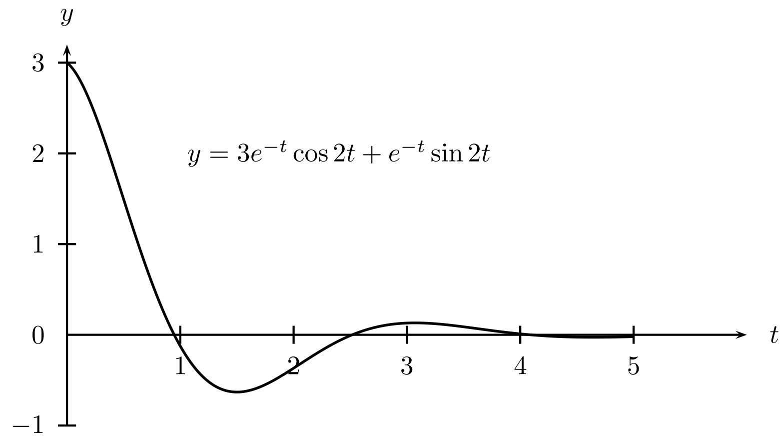
Hence, $c_1 = 3$ and $c_2 = 1$, so that the solution of the initial value problem is

$$\boxed{y = 3e^{-t}\cos 2t + e^{-t}\sin 2t.}$$

To answer the rest of the question, note that the characteristic polynomial has a pair of complex conjugate roots. Hence, the equation represents an **underdamped** system. To find the first $t > 0$ such that $y(t) = 0$, note that

$$\begin{aligned} y(t) = 0 &\iff 3\cos 2t + \sin 2t = 0 \\ &\iff \tan 2t = -3 \\ &\iff 2t \approx 1.8925 + \pi k \text{ for some integer } k \\ &\iff t \approx .9462 + \frac{\pi}{2}k \text{ for some integer } k. \end{aligned}$$

The smallest such positive t is obtained for $k = 0$ and hence the first time that $y(t) = 0$ for $t > 0$ occurs for $t \approx .9462$.



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