Exam 4 will be on Tuesday, November 24. The syllabus is Sections 9.1 to 9.5 from the text, and Sections 10.1 to 10.4 from the Fourier series supplement. You should review the assigned exercises in these sections. Following is a brief list (not necessarily complete) of terms, skills, and formulas with which you should be familiar.

- For a constant coefficient homogeneous linear system

  $$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \boldsymbol{y}_0,$$

  the unique solution is

  $$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}_0,$$

  where the matrix exponential e^{At} is *defined* by the infinite series

  $$e^{At} = I_n + At + \frac{1}{2}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots + \frac{1}{n!}A^nt^n + \cdots .$$

  Know how to compute e^{At} from the definition for some simple 2×2 matrices, as is done in the examples on Pages 650–1.

- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, is a 2×2 matrix then the system of differential equations

  $$\boldsymbol{y}' = A\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix},$$

  which can also be written as

  $$\begin{array}{rcl} y_1' & = & ay_1 + by_2 \\ y_2' & = & cy_1 + dy_2 \end{array} \qquad y_1(0) = c_1,\ y_2(0) = c_2,$$

  has the solution

  $$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0).$$

  Since

  $$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\},$$

  (see Theorem 4, Page 652) the following algorithm can be used to solve $\boldsymbol{y}' = A\boldsymbol{y}$ when the matrix A is a constant 2×2 matrix.

  Algorithm 1. 1. Form the matrix $sI - A = \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix}$.

  2. Compute the characteristic polynomial

  $$p(s) = \det(sI - A) = \det \begin{bmatrix} s-a & -b \\ -c & s-d \end{bmatrix} = s^2 - (a+d)s + (ad - bd).$$

  3. Compute

  $$(sI - A)^{-1} = \frac{1}{p(s)} \begin{bmatrix} s-d & b \\ c & s-a \end{bmatrix} = \begin{bmatrix} \dfrac{s-d}{p(s)} & \dfrac{b}{p(s)} \\ \dfrac{c}{p(s)} & \dfrac{s-a}{p(s)} \end{bmatrix}.$$

4. Compute

$$\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} \mathcal{L}^{-1}\left\{\dfrac{s-d}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{b}{p(s)}\right\} \\ \mathcal{L}^{-1}\left\{\dfrac{c}{p(s)}\right\} & \mathcal{L}^{-1}\left\{\dfrac{s-a}{p(s)}\right\} \end{bmatrix} = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix}.$$

5. The solution $\boldsymbol{y}(t)$ is then

$$\boldsymbol{y}(t) = e^{At}\boldsymbol{y}(0) = \left(\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\}\right)\boldsymbol{y}(0) = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1h_1(t)+c_2h_2(t) \\ c_1h_3(t)+c_2h_4(t) \end{bmatrix}.$$

- Know Fulmer's method for computing e^{At} as described in Section 9.4, Page 660.
- Know the solution algorithm for constant coefficient first order systems. See Page 673, or Theorem 3, Page 666.
- Know what it means for a function to be periodic of period p.
- Know how to recognize if a function is even or odd (or neither).
- Know how to compute the Fourier series of a periodic function by use of the Euler-Fourier formulas. (Page 733, Supplement).
- Know the conditions under which a Fourier series converges to $f(t)$: Theorem 2, page 748 of the Supplement.
- Know how to compute the Fourier sine and cosine series of a function $f(t)$ defined on an interval $0 \le t \le L$.

The following is a small set of exercises of types identical to those already assigned.

1. Let $A = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix}$

   (a) Compute $(sI - A)$ and $(sI - A)^{-1}$.

   (b) Find $\mathcal{L}^{-1}((sI - A)^{-1})$.

   (c) What is e^{At}?

   (d) Solve the system $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

2. Solve the matrix differential equation $\boldsymbol{y}' = A\boldsymbol{y}$ where $A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$.

3. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

4. Solve the initial value problem:

$$\boldsymbol{y}' = \begin{bmatrix} 0 & -3 \\ 3 & 6 \end{bmatrix} \boldsymbol{y}, \quad \boldsymbol{y}(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$.

   (a) Compute A^2, A^3, etc.

   (b) Compute e^{At} from the definition as a series.

   (c) Solve the initial value problem $\boldsymbol{y}' = A\boldsymbol{y}$, $\boldsymbol{y}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

6. Let

$$f(t) = \begin{cases} 0 & \text{if } -\pi < t < 0 \\ t & \text{if } 0 \le t < \pi \end{cases} \qquad f(t + 2\pi) = f(t).$$

   Sketch the graph of f and compute the Fourier series of f.

7. Let $f(t) = \begin{cases} t^2 & \text{if } 0 < t < 2 \\ t - 4 & \text{if } 2 < t < 4 \end{cases}$.

   (a) Let f_1 be the periodic extension of $f(t)$ to all of $\mathbb{R}$, assuming a period of 4. Sketch the graph of f_1 on $[-4, 4]$. What is the natural way to define f_1 at 2 so that the Fourier series of f_1 converges to f_1?

   (b) Sketch the graph of the odd extension f_2 of f to the interval $[-4, 4]$.

   (c) Sketch the graph of the even extension f_3 of f to the interval $[-4, 4]$.

8. Let $f(t)$ be a periodic function of period π. The formula for $f(t)$ on the interval $[0, \pi)$ is

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi/4 \\ 5 & \text{if } \pi/4 \le t < 3\pi/4 \\ 0 & \text{if } 3\pi/4 \le t < \pi \end{cases}.$$

   (a) Sketch the graph of $f(t)$ on the interval $[-2\pi, 2\pi]$.

   (b) Compute the Fourier series of $f(t)$.

   (c) Let $g(t)$ be the sum of the Fourier series found in part (b). Compute $g(0)$, $g(\pi/4)$, and $g(4)$.

9. Compute the Fourier cosine series of the function

$$f(t) = 1 - t, \qquad 0 < t < \pi.$$

Answers

1. $sI - A = \begin{bmatrix} s-1 & 2 \\ 3 & s-2 \end{bmatrix}$; $(sI-A)^{-1} = \begin{bmatrix} \dfrac{s-2}{(s-4)(s+1)} & \dfrac{-2}{(s-4)(s+1)} \\ \dfrac{-3}{(s-4)(s+1)} & \dfrac{s-1}{(s-4)(s+1)} \end{bmatrix}$

(b) $\frac{1}{5}\begin{bmatrix} 2e^{4t}+3e^{-t} & -2e^{4t}+2e^{-t} \\ -3e^{4t}+3e^{-t} & -3e^{4t}+2e^{-t} \end{bmatrix}$ (c) e^{At} is same as $\mathcal{L}^{-1}((sI-A)^{-1})$.

(d) $\boldsymbol{y}(t) = \frac{1}{5}\begin{bmatrix} -8e^{4t}+3e^{-t} \\ -6e^{4t}+3e^{-t} \end{bmatrix}$

2. $\boldsymbol{y}(t) = \frac{1}{6}\begin{bmatrix} (5c_1-c_2)e^{4t}+(c_1+c_1)e^{-2t} \\ (-5c_1+c_2)e^{4t}+(5c_2+5c_1)e^{-2t} \end{bmatrix}$

3. $\boldsymbol{y}(t) = \frac{1}{2}\begin{bmatrix} 1+e^{4t} \\ -2+2e^{4t} \end{bmatrix}$

4. $\boldsymbol{y}(t) = \begin{bmatrix} e^{3t}+3te^{3t} \\ -2e^{3t}-3te^{3t} \end{bmatrix}$

5. (a) $A^2 = A^3 = \cdots = A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all $n \geq 2$.

   (b) $e^{At} = I + At = \begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix}$

   (c) $\boldsymbol{y}(t) = \begin{bmatrix} 3-2t \\ -2+t \end{bmatrix}$.

6.
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_0^{\pi} t\,dt = \frac{1}{\pi}\frac{t^2}{2}\bigg|_0^{\pi} = \frac{\pi}{2}.$$

For $n \geq 1$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\cos nt\,dt = \frac{1}{\pi}\left[\frac{\cos nt}{n^2} + \frac{t}{n}\sin nt\right]_0^{\pi} = \frac{\cos n\pi - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi} t\sin nt\,dt = \frac{1}{\pi}\left[\frac{\sin nt}{n^2} - \frac{t}{n}\cos nt\right]_0^{\pi} = -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}.$$

Thus,

$$f(t) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{\pi n^2}\cos nt + \frac{(-1)^{n+1}}{n}\sin nt\right).$$

7. (a) $f_1(2) = \dfrac{f(2^+) + f(2^-)}{2} = \dfrac{4 + (-2)}{2} = 1$

8. (a)

   (b) $f(t)$ is even so we only need to compute a_n. The period is π so $l = \pi/2$. Then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt = \frac{2}{\pi}\int_{\pi/4}^{3\pi/4} 5\,dt = 5$$

and for $n \geq 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos 2nt\,dt = \frac{2}{\pi}\int_{\pi/4}^{3\pi/4} 5\cos 2nt\,dt = \frac{10}{2n\pi}\sin 2nt\Big|_{\pi/4}^{3\pi/4}$$

$$= \frac{5}{\pi n}\left(\sin\frac{6n\pi}{4} - \sin 2n\pi 4\right) = \begin{cases} 0 & n = 2k, \\ -\frac{10}{n\pi} & n = 1, 5, 9, \ldots, \\ \frac{10}{n\pi} & n = 3, 7, 11, \ldots. \end{cases}$$

Thus,

$$f(t) \sim \frac{5}{2} + \frac{10}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{\cos 2(2n-1)t}{2n-1}.$$

(c) $g(0) = 0$, $g(\pi/4) = 5/2$, $g(4) = g(4-\pi) = g(.8584) = 5$

9.

$$f(t) \sim \left(1 + \frac{\pi}{2}\right) + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nt.$$