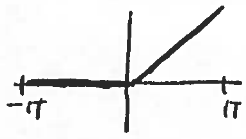
Math 2020 Exam V Solutions.

1.



$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big]_0^{\pi} = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{1}{\pi}\left[\frac{\cos nx}{n^2} + \frac{x}{n}\sin nx\right]_0^{\pi}$$

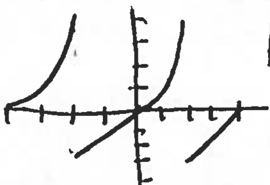
$$= \frac{1}{\pi n^2}(\cos n\pi - 1) = \frac{(-1)^n - 1}{\pi n^2}.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n^2} - \frac{x}{n}\cos nx\right]_0^{\pi}$$

$$= -\frac{\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}.$$

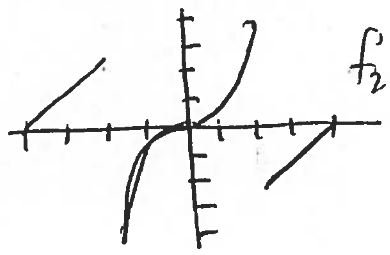
$$\Rightarrow f(x) \sim \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{\pi n^2}\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right].$$

2.

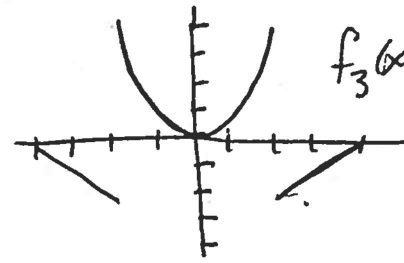


$f_1(x)$.

Define $f_1(2) = \frac{4-2}{2} = 1.$



$f_2(x)$



$f_3(x)$

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3. (a) $g(x) = x = \frac{1}{2}f'(x) = \frac{1}{2}\left(4\sum_{n=1}^{\infty}\frac{d}{dx}\left((-1)^n\frac{\cos nx}{n^2}\right)\right) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}.$

(b) $x^3 = 3\int_0^x f(t)\,dt = 3\left[\frac{\pi^2}{3}x + 4\sum_{n=1}^{\infty}(-1)^n\int_0^x \frac{\cos nt\,dt}{n^2}\right]$

$$= \pi^2 x + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx = 2\pi^2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n} + 12\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nx$$

$$= \sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12(-1)^n}{n^3}\right)\sin nx$$

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4. $y_2'' + 2y_2' + 3y_2 = 5\sin 2x.$

$$y_2 = A_2\cos 2x + B_2\sin 2x$$
$$\Rightarrow y_2' = -2A_2\sin 2x + 2B_2\cos 2x$$
$$y_2'' = -4A_2\cos 2x - 4B_2\sin 2x$$

$$\Rightarrow (-A_2 + 4B_2)\cos 2x + (-B_2 - 4A_2)\sin 2x = 5\sin 2x$$

$$\Rightarrow \begin{array}{l} -A_2 + 4B_2 = 0 \\ -4A_2 - B_2 = 5 \end{array} \Rightarrow A_2 = \frac{-20}{17},\ B_2 = \frac{-5}{17}.$$

$$\Rightarrow y_2 = \frac{-20}{17}\cos 2x - \frac{5}{17}\sin 2x.$$